Spontaneous Symmetry Breaking and Goldstone’s Theorem

- **Exact symmetry**: \([T^i, \mathcal{L}] = 0\) (equations of motion); \(T^i|0\rangle = 0\) (vacuum)

- **Explicit breaking**: \([T^i, \mathcal{L}] \neq 0\), e.g., \(\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1\) with \([T^i, \mathcal{L}_1] \neq 0\)

- **Spontaneous breaking**: \(T^i|0\rangle \neq 0\)

- **Coleman’s theorem**: explicit breaking induces spontaneous

- **The Goldstone alternative**: \([T^i, \mathcal{L}] = 0\) allows either
  - **Symmetry unbroken** (Wigner-Weyl realization): \(T^i|0\rangle = 0\), or
  - **SSB**: \(T^i|0\rangle \neq 0 \Rightarrow\) **massless Nambu-Goldstone boson** (or Higgs mechanism for gauge symmetry)
Single Hermitian Field

- No continuous symmetries; can impose discrete $Z_2 \ (\phi \rightarrow -\phi)$:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - V (\phi), \quad V (\phi) = \frac{\mu^2 \phi^2}{2} + \frac{\lambda \phi^4}{4}$$

$$\left( \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \phi = -\frac{\partial V}{\partial \phi} = - [\mu^2 + \lambda \phi^2] \phi$$

- Lowest energy solution for classical field: $\phi_{class} \equiv \langle 0 | \phi | 0 \rangle \equiv \langle \phi \rangle$ (vacuum expectation value [VEV])

  - $\langle 0 | \phi | 0 \rangle = \text{constant in } x$, minimizes $V$

    $$\frac{\partial V}{\partial \phi} \bigg|_{\langle \phi \rangle} = 0, \quad \frac{\partial^2 V}{\partial \phi^2} \bigg|_{\langle \phi \rangle} > 0$$

  - Require $\lambda > 0 \ (V \text{ bounded below}); \ \mu^2 \ arbitrary$
\[ V(\phi) \]

- \( \mu^2 > 0 \): minimum at \( \langle 0|\phi|0 \rangle = 0 \), symmetry unbroken
- \( \mu^2 < 0 \): \( \langle 0|\phi|0 \rangle = 0 \) is unstable; minima at \( \pm \nu \equiv \pm \sqrt{-\mu^2/\lambda} \) 
  \( \phi \to -\phi \) symmetry spontaneously broken
- Define \( \phi = \nu + \phi' \); \( \phi' \) is ordinary quantum field \( \langle 0|\phi'|0 \rangle = 0 \)

\[ L(\phi) = L(\nu + \phi') = \frac{1}{2} (\partial_\mu \phi')^2 - V(\phi') \]

\[ V(\phi') = \frac{-\mu^4}{4\lambda} \cosm\,\text{const.} - \mu^2 \phi'^2 + \lambda \nu \phi'^3 + \frac{\lambda}{4} \phi'^4 \]

**cosm. const.** \( \mu^2 \phi' = -2\mu^2 > 0 \) **induced cubic**
Can add explicit $\mathbb{Z}_2$-breaking terms ($\phi$ or $\phi^3$), e.g.,

$$V(\phi) = \frac{\mu^2 \phi^2}{2} - a \phi + \frac{\lambda \phi^4}{4}, \quad a > 0$$

$\Rightarrow \langle 0 | \phi | 0 \rangle \neq 0$, even for $\mu^2 > 0$ (Coleman’s theorem)

- For $\mu^2 > 0$ and $a$ small: $\nu = \langle \phi \rangle = a/\mu^2 + \mathcal{O}(a^3)$

$$V(\phi') = -\frac{a^2}{2\mu^2} + \frac{\mu^2}{2} \phi'^2 + \lambda \nu \phi'^3 + \frac{\lambda}{4} \phi'^4$$

- For $\mu^2 < 0$: global (true) minimum at $\nu = \nu_0 + \frac{a}{2\nu_0^2} + \mathcal{O}(a^2)$
A Complex Scalar

- Complex scalar \((\lambda > 0)\):

\[
\mathcal{L}_0 = (\partial_\mu \phi)^\dagger \partial^\mu \phi - V(\phi), \quad V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2
\]

- continuous global \(U(1)\) symmetry, \(\phi \rightarrow e^{i\beta} \phi\)

- Hermitian basis: \(\phi = (\phi_1 + i\phi_2)/\sqrt{2} \Rightarrow O(2)\) symmetry

\[
\mathcal{L}_0 = \frac{1}{2} \left[ (\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 \right] - V(\phi_1, \phi_2), \quad V = \frac{\mu^2}{2} (\phi_1^2 + \phi_2^2) + \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2
\]

\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} \rightarrow \begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix} \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
\]

(rotation; \(U(1)\) and \(SO(2)\) equivalent)
\( \mu^2 > 0: \text{ minimum at } \nu_1 = \nu_2 = 0 \) (Wigner-Weyl realization);
degenerate \( \phi_{1,2} \) (or \( \phi, \phi^\dagger \)), conserved charge, quartics related

– Can add explicit breaking

\[
\mathcal{L} = \mathcal{L}_0 - \frac{\epsilon}{2} \phi_2^2 \\
m_1^2 = \mu^2, \quad m_2^2 = \mu^2 + \epsilon
\]
• $\mu^2 < 0$ and $\epsilon = 0$ (Nambu-Goldstone realization): degenerate minima of Mexican hat potential along

$$\phi_1^2 + \phi_2^2 = \nu^2 \equiv \frac{-\mu^2}{\lambda} > 0$$

– Choose axes so that $\phi_1 = \nu + \phi'_1$, $\phi_2 = \phi'_2$:

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi'_1 \right)^2 + \frac{1}{2} \left( \partial_\mu \phi'_2 \right)^2 - V (\phi'_1, \phi'_2)$$

$$V = \frac{-\mu^4}{4\lambda} - \mu^2 \phi'_1^2 + \lambda \nu \phi'_1 (\phi'_1^2 + \phi'_2^2) + \frac{\lambda}{4} (\phi'_1^2 + \phi'_2^2)^2$$

– $m_1^2 = -2\mu^2 > 0$ and $m_2^2 = 0$ (Nambu-Goldstone boson)

– Can prove for any SSB of continuous global symmetry
• Add small explicit breaking $-\epsilon \phi_2^2/2 \Rightarrow$ unique vacuum (up to sign), with
  \[ m_1^2 = -2\mu^2, \quad m_2^2 = \epsilon \ll m_1^2 \]

• $\phi_2$ is pseudo-Goldstone boson (e.g., pions in QCD)
Spontaneously Broken Chiral Symmetry

- Chiral fermion $\psi = \psi_L + \psi_R$ (no mass term) and complex scalar $\phi$:

$$\mathcal{L} = \bar{\psi}_L i \not{\partial} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R - h \bar{\psi}_L \psi_R \phi - h \bar{\psi}_R \psi_L \phi^\dagger + (\partial_\mu \phi)^\dagger \partial^\mu \phi - V(\phi)$$

with

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

- Chiral symmetry:

$$\phi \rightarrow e^{i\beta} \phi, \quad \psi_L \rightarrow \psi_L, \quad \psi_R \rightarrow e^{-i\beta} \psi_R$$

- For $\mu^2 < 0$ (and $\lambda > 0$): $\phi_1 = \nu + \phi_1', \quad \phi_2 = \phi_2'$

$$\mathcal{L}_{Yuk} = -\frac{h \nu}{\sqrt{2}} \bar{\psi} \psi \left(1 + \frac{\phi_1'}{\nu}\right) - \frac{h}{\sqrt{2}} i \bar{\psi} \gamma^5 \psi \phi_2'$$

$$\left(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L = \bar{\psi} \psi \text{ and } \bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L = \bar{\psi} \gamma^5 \psi\right)$$
- Massless Goldstone boson $\phi'_2$
- Effective $\psi$ mass: $m_\psi = \frac{h\nu}{\sqrt{2}}$
- Scalar (pseudoscalar) couplings of $\phi_1(\phi_2)$, strength $\frac{h}{\sqrt{2}} = m_\psi/\nu$
# Possibilities for Continuous Symmetry

## Exact Lagrangian Symmetry \([U_G, \mathcal{L}] = 0\)

| \(U_G|0\rangle = |0\rangle\) | \(U_G|0\rangle \neq |0\rangle\) |
|---|---|
| exact symmetry<br>(Wigner-Weyl) | spontaneous symmetry breaking<br>(Nambu-Goldstone) |

- degenerate multiplets
- conserved charges
- relations between interactions
- chiral: massless fermions
- gauge: massless gauge bosons

## Explicit breaking \([U_G, \mathcal{L}] \neq 0\) (global only)

- multiplet splitting, etc.
- chiral: fermions acquire mass

- Goldstone bosons acquire mass

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