Quantum Chaos and Effective Field Theory

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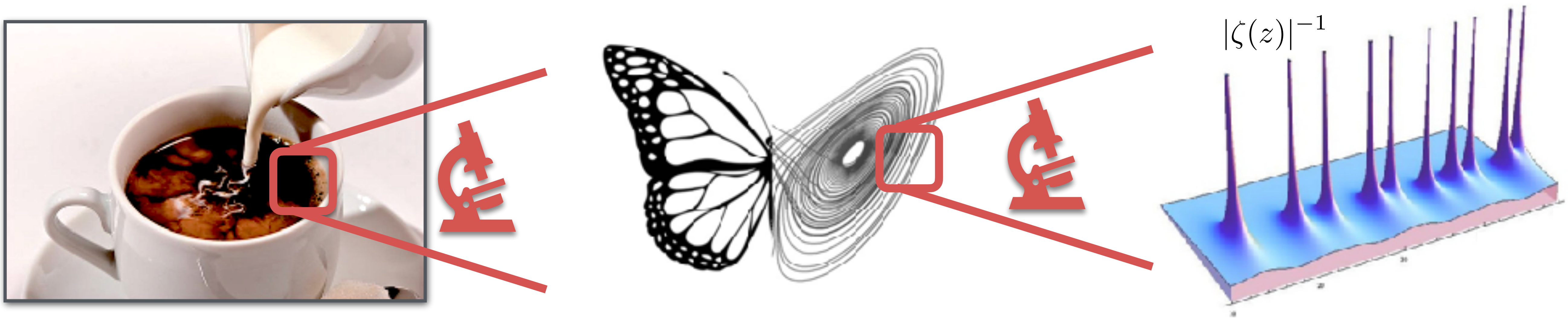
Strings Seminar, UBC Vancouver

2301.05698 with C. Choi, M. Mezei, G. Sarosi

Thermalization & Chaos

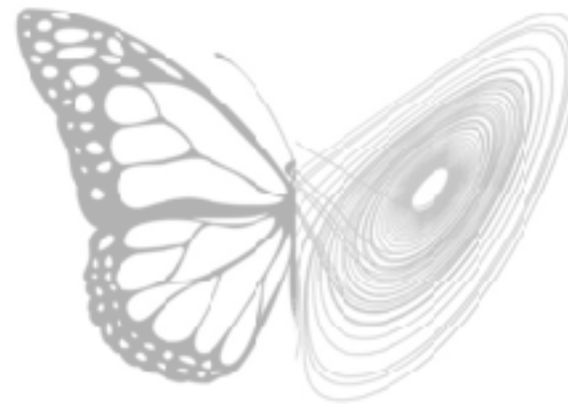
- Many different manifestations on different scales:



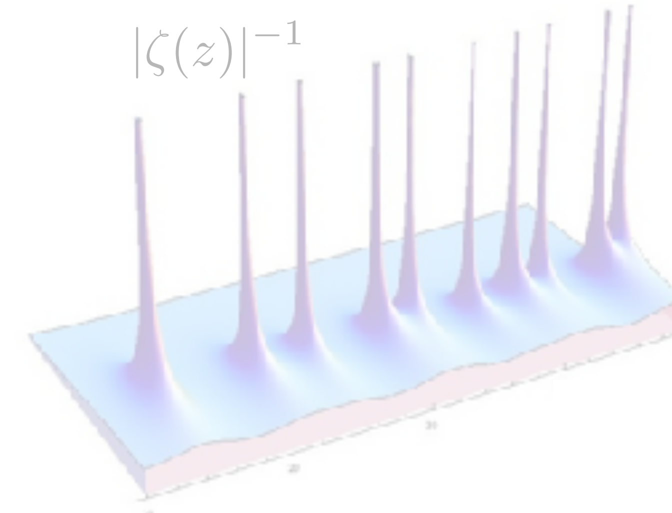
transport & dissipation | **quantum butterfly effect** | **random matrix universality**

- Wanted: universal features, fundamental constraints, interconnections
- Via holography: all related to black holes & quantum gravity

transport & dissipation



quantum butterfly effect



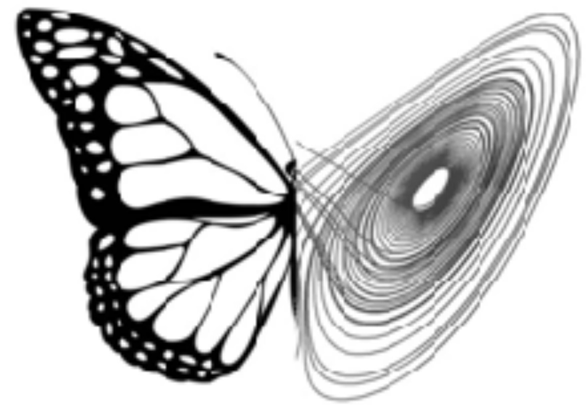
random matrix universality

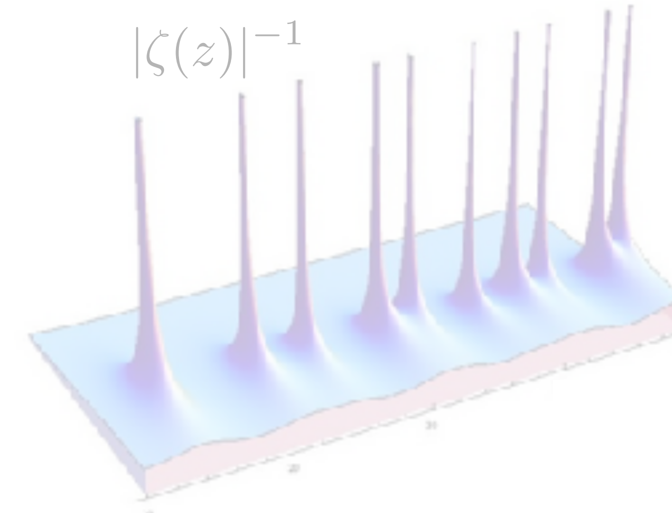
Hydrodynamics: paradigm for EFT capturing thermal physics in a universal way

- Physics of ‘slow’ relaxation of conserved quantities via transport

hydrodynamics of strongly coupled QFT ⟷ (AdS/CFT duality) ⟷ black hole fluctuations & relaxation to equilibrium

[Bhattacharyya/Hubeny/Minwalla/Rangamani ’07]



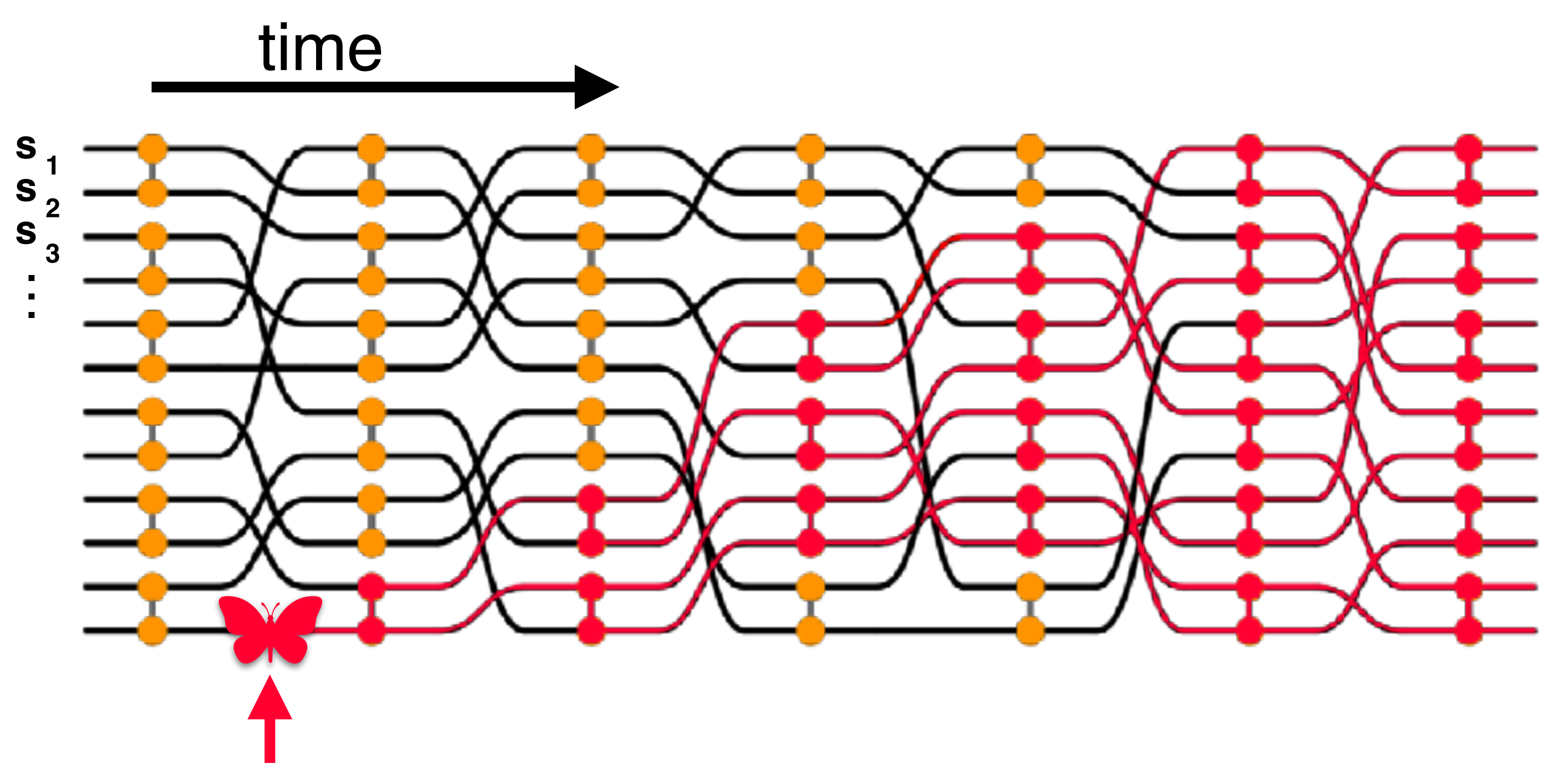


transport & dissipation

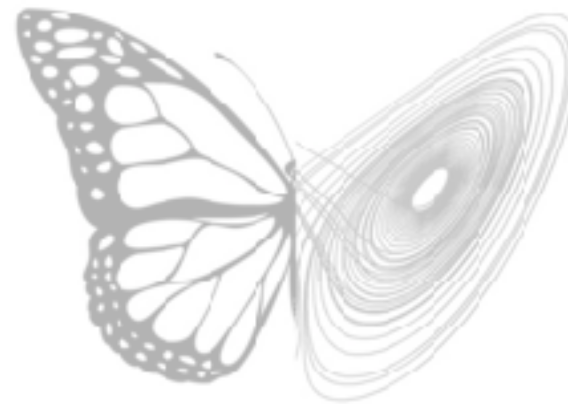
quantum butterfly effect

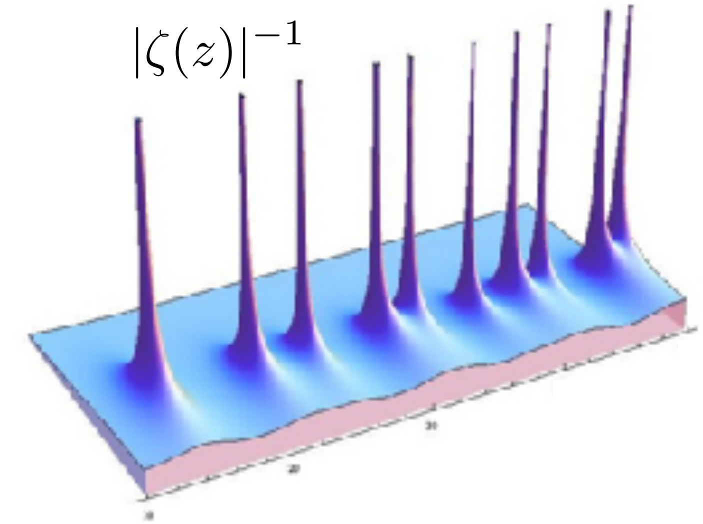
random matrix universality

Exp. growth of perturbation in strongly interacting quantum systems (“operator growth”)



- Related to momentum increase of a probe falling into a black hole, shockwave scattering, …





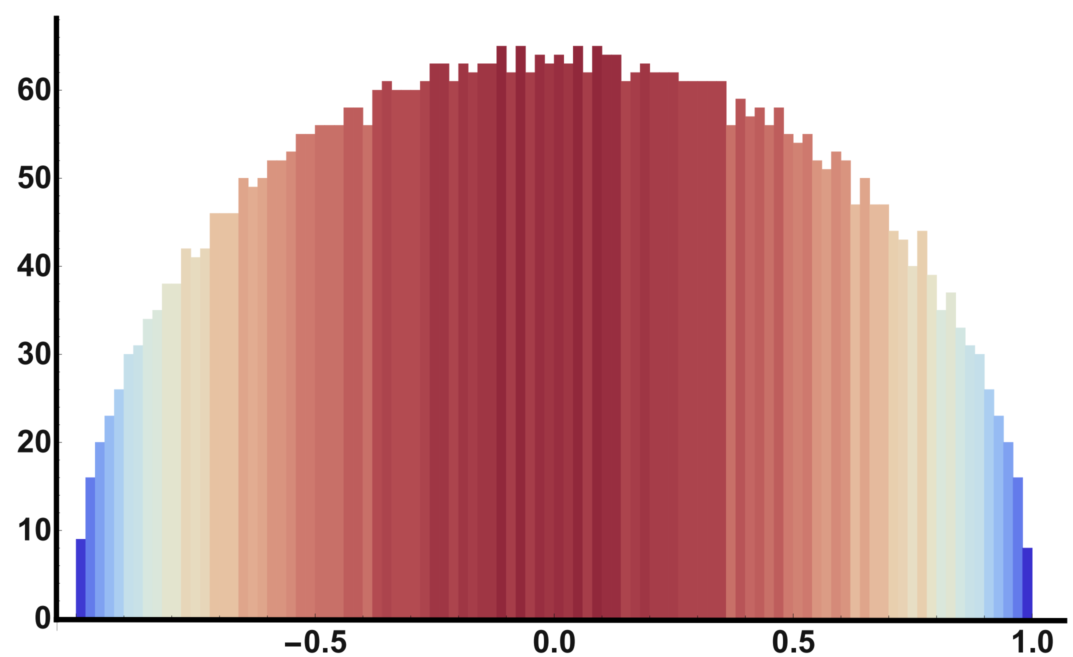
transport & dissipation

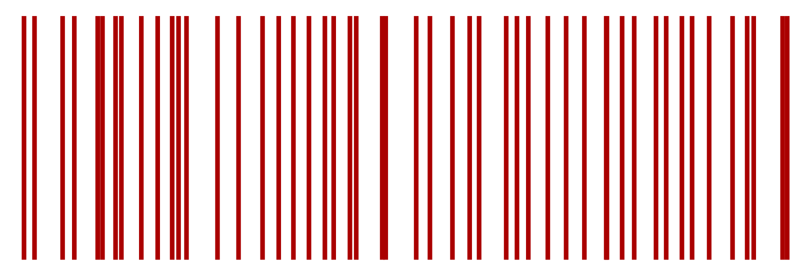
quantum butterfly effect

random matrix universality

- RMT ⟶ statistics of energy level spacings of chaotic quantum systems

[Wigner '56] ...

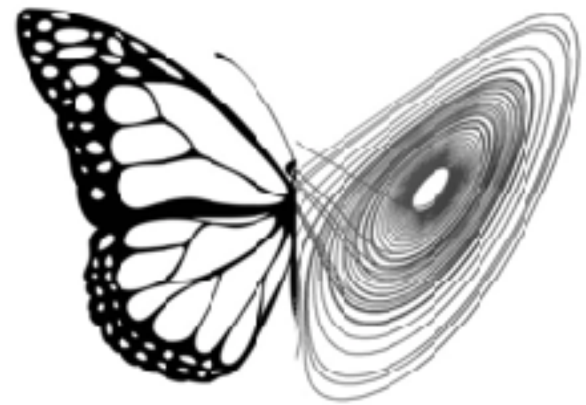




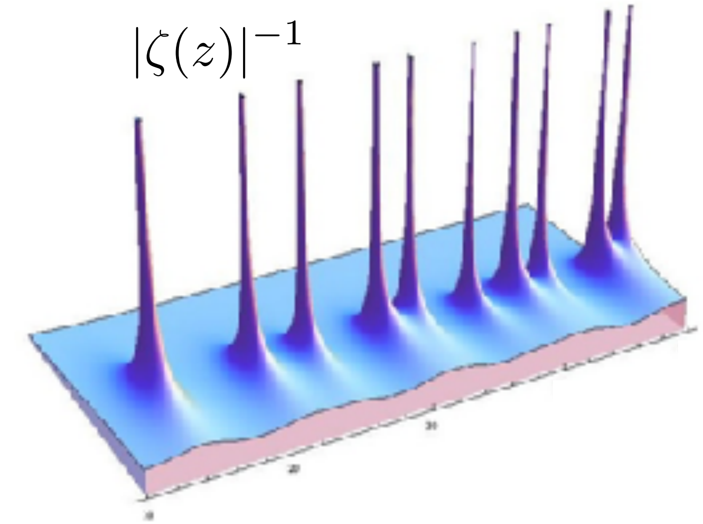
- AdS/CFT relates spectral correlations to spacetime wormholes, …

[Saad/Shenker/Stanford '18]

transport & dissipation



quantum butterfly effect



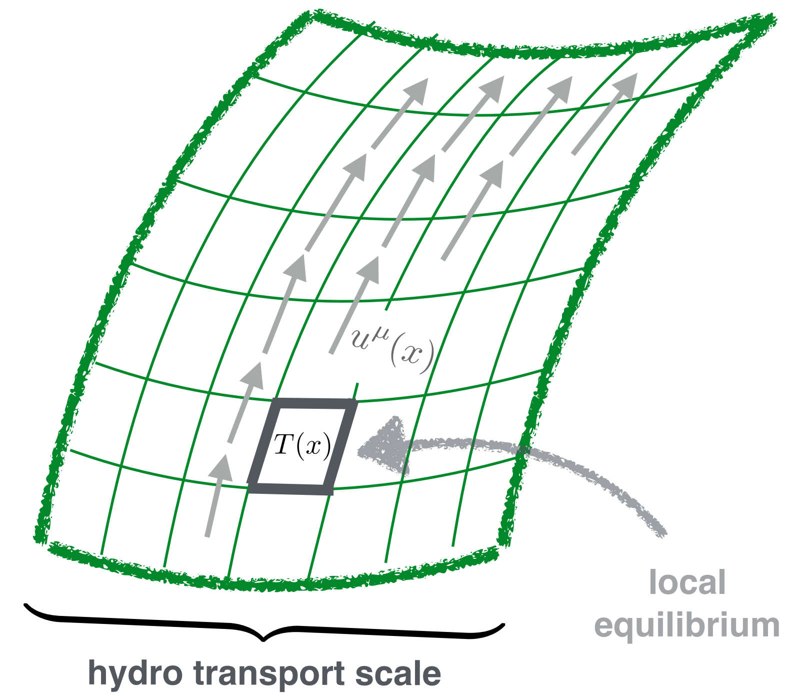
random matrix universality

- Different manifestations of chaos, all related to gravity & black holes
- **Effective field theory:** framework to capture universal aspects
  - **EFT of classical and fluctuating hydrodynamics:** a lot of progress on identifying symmetries and effective actions

    [FH/Loganayagam/Rangamani] [Crossley/Glorioso/Liu] [Jensen/Pinzani-Fokeeva/Yarom]
  - This talk: **EFT of quantum chaos**

EFT for quantum chaos

- Hydrodynamics:
  - EFT for correlators of $T^{\mu\nu}$
  - Degrees of freedom: embedding maps of fluid elements into spacetime (“Lagrangian description”)
  - Systematic long wavelength expansion



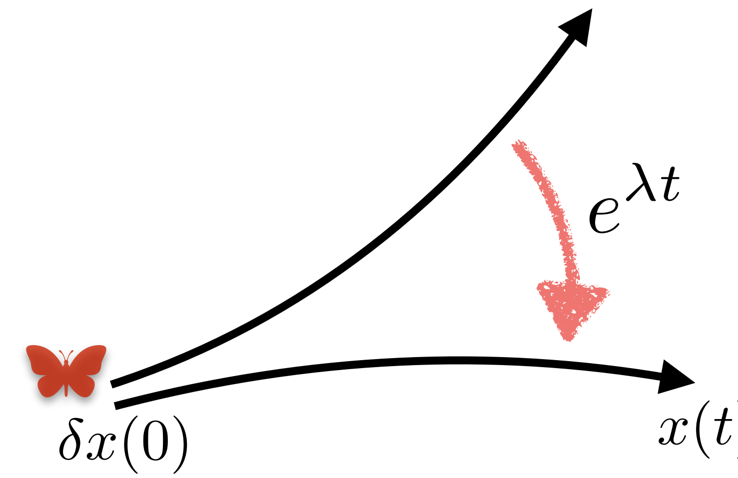
- Quantum chaos:
  - What correlators?
  - What effective degrees of freedom?
  - Expansion in what?

Outline

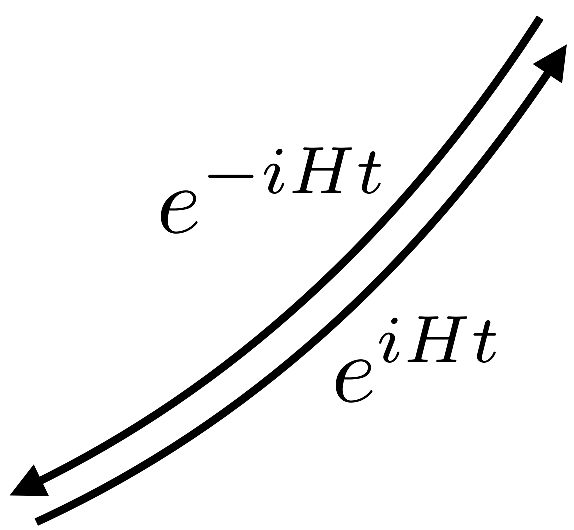
- Introduction
- Quantum butterfly effect & EFT of chaos
- Sub-maximal chaos in large q SYK
- Large q SYK chain
- Conclusion

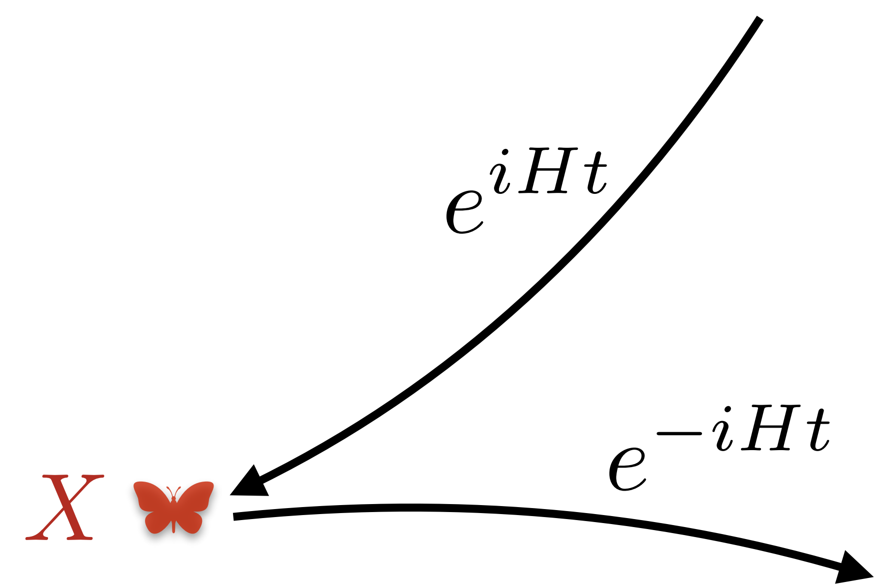
Quantum Butterfly Effect

Quantum Butterflies



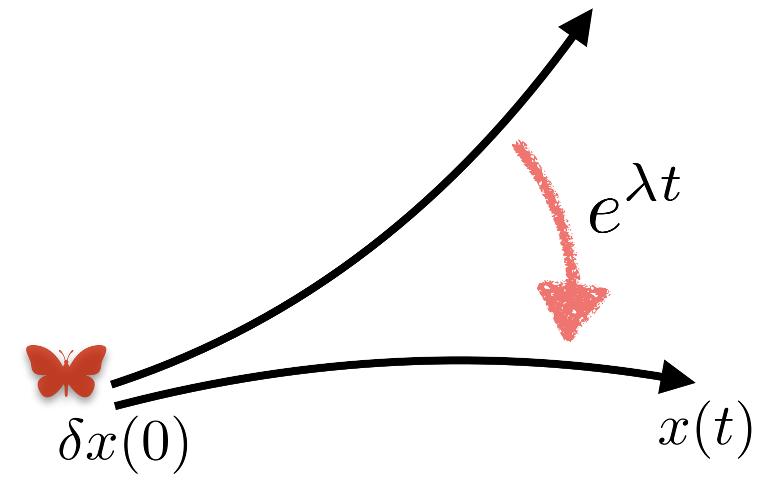
- **Classical** butterfly effect:
  exponential sensitivity to initial condition
- **Quantum** butterfly effect:
  $X(t) = e^{iHt} X \, e^{-iHt}$ is 'complicated' even if X was 'simple'





$$e^{iHt} X e^{-iHt} = X + it[H, X] - \frac{t^2}{2}[H, [H, X]] + \dots$$

Quantum Butterflies



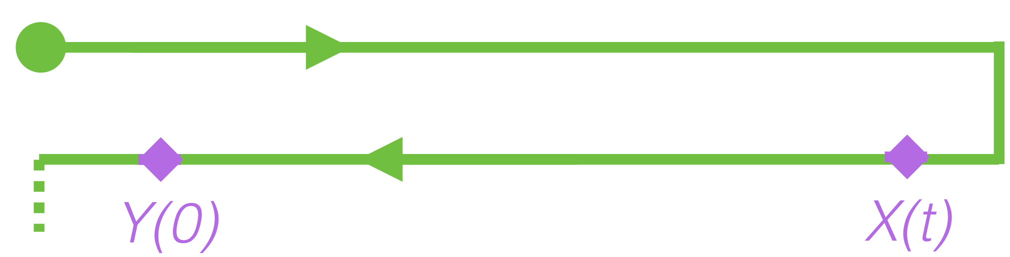
- **Classical** butterfly effect:
  exponential sensitivity to initial condition

- **Quantum** butterfly effect:
  $X(t) = e^{iHt} X\, e^{-iHt}$ is ‘complicated’ even if X was ‘simple’

- To quantify this, compare the following states:

$|X(t)Y(0)\rangle$:

$|Y(0)X(t)\rangle$:



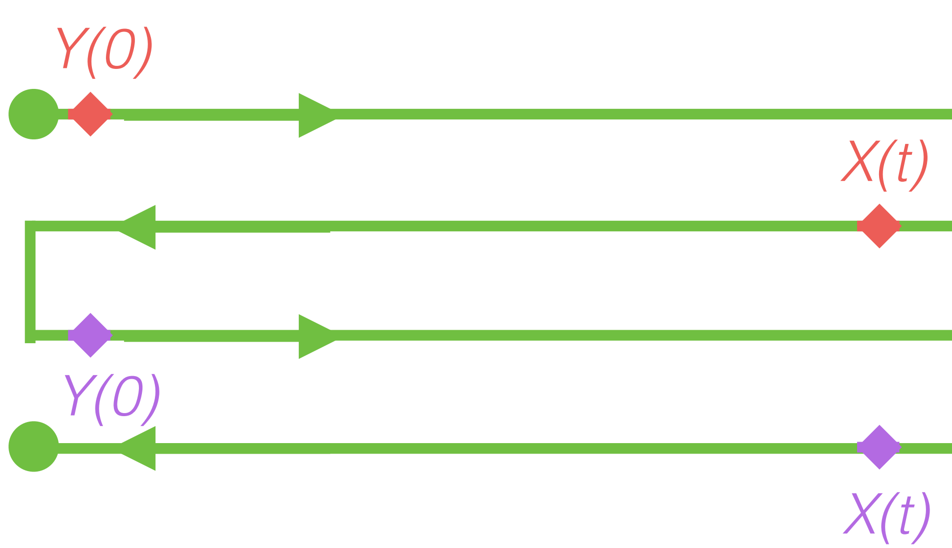
- Overlap of $|X(t)Y(0)\rangle$ and $|Y(0)X(t)\rangle$:

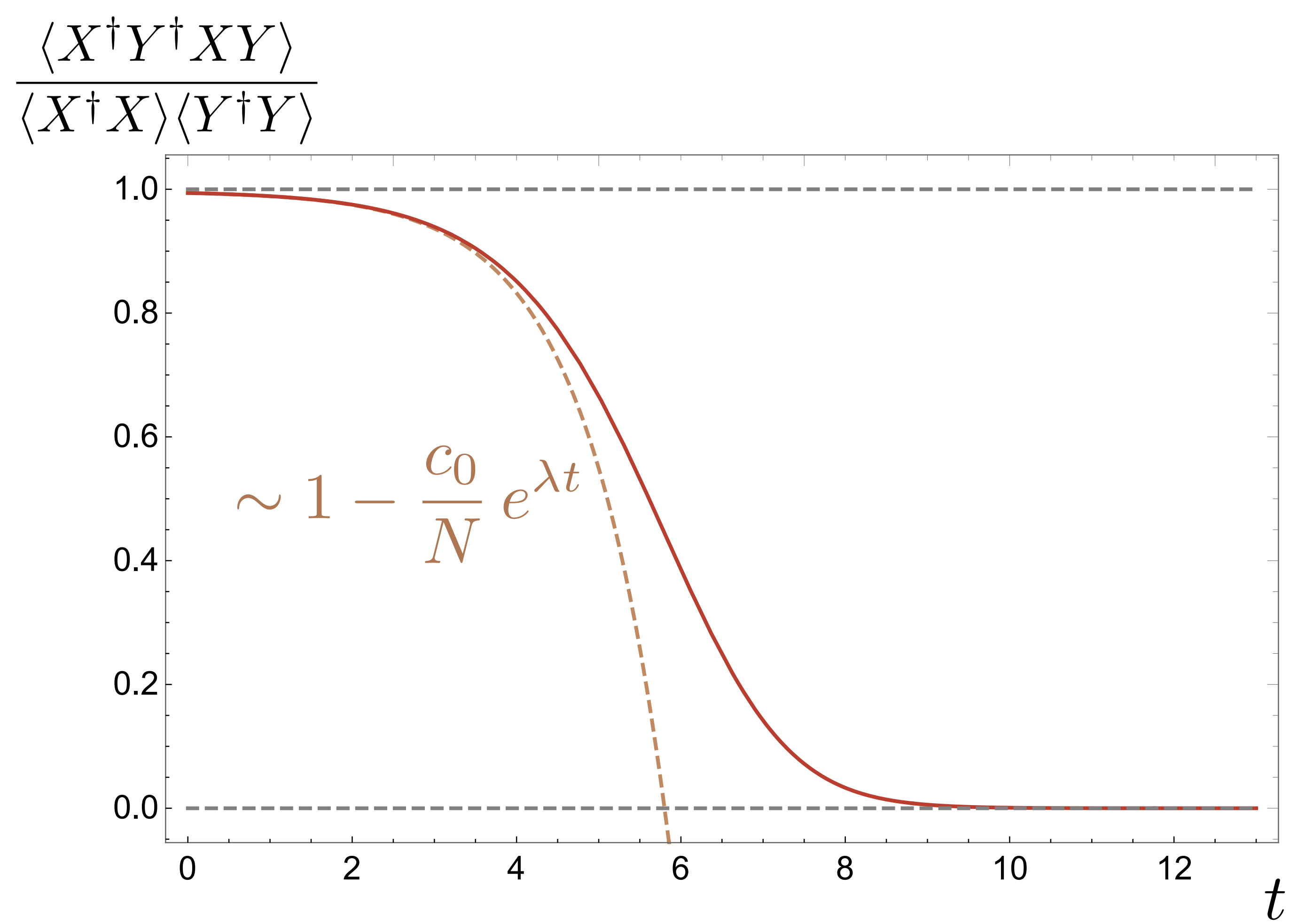
$$\langle X^\dagger(t)Y^\dagger(0)|X(t)Y(0)\rangle_\beta \sim \langle X^\dagger X\rangle\langle Y^\dagger Y\rangle \left(1 - \frac{c_0}{N}\, e^{\lambda t} + \ldots\right)$$

Out-of-time-order correlation function (OTOC)

—> Lyapunov exponent λ : measure of quantum chaos

[Larkin/Ovchinnikov '68] [Kitaev '14] [Shenker/Stanford '14]





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Out-of-time-order correlation function (OTOC)

—> Lyapunov exponent λ : measure of quantum chaos

[Larkin/Ovchinnikov '68] [Kitaev '14] [Shenker/Stanford '14]

$$\left(\frac{1}{N}\, e^{\lambda t}\right)^k \sim \mathcal{O}(1)$$

"Ruelle"

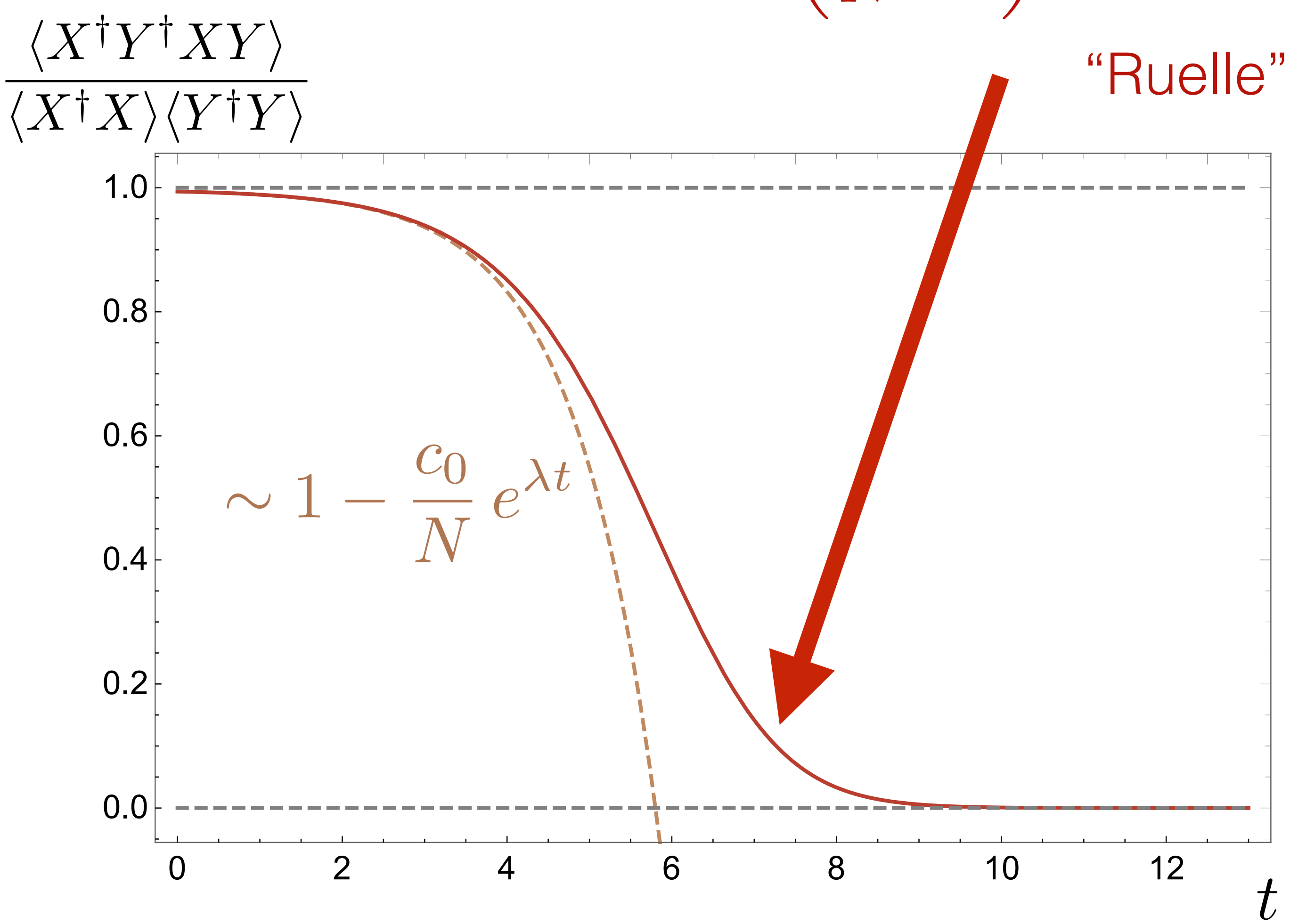
- Large butterfly effect at **scrambling time:**

$$t_* \sim \frac{1}{\lambda} \log N$$

- **Chaos bound:**

$$\lambda \leq 2\pi T$$

[Maldacena/Shenker/Stanford '16]



EFT for maximal chaos

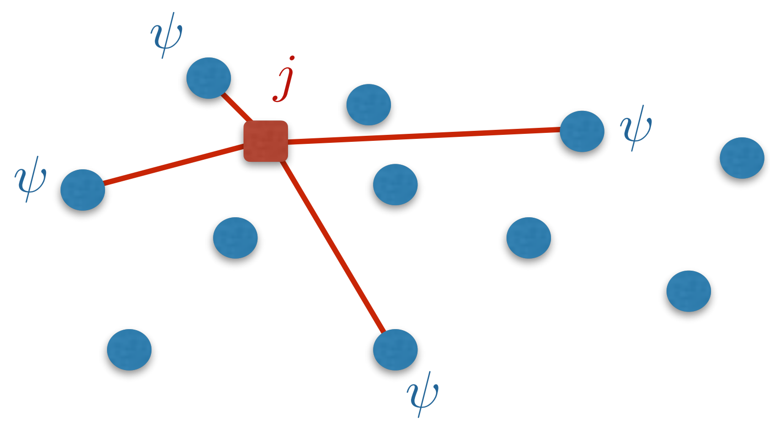
SYK model

- N Majorana fermions with random, Gaussian couplings

$$H = -\sum_{ijkl}^{N} j_{ijkl}\, \psi_i \psi_j \psi_k \psi_l$$

$$\overline{j_{ijkl}} = 0\,, \quad \overline{j_{ijkl}^2} = J^2/N^3$$

- Solvable for $N \gg \beta J \gg 1$



- 'Mean field' description at large N in terms of bilocal 2-point function

$$G(\tau,\tau') = \frac{1}{N}\sum_{i=1}^{N} \langle \psi_i(\tau)\psi_i(\tau')\rangle$$

- $\beta J \gg 1$: $S_{\text{eff}}[G]$ is approximately $\text{diff}(S^1)$ invariant: $\tau \to f(\tau)$
  - The saddle point solution breaks $\text{diff}(S^1) \to SL(2,\mathbb{R})$:

$$G_c(\tau-\tau') \propto \frac{1}{(\tau-\tau')^{2/q}}$$

[Sachdev/Ye '93] [Kitaev '15]
[Maldacena/Stanford '16] ...

- The pseudo-Goldstone associated with reparametrizations $\tau \to f(\tau)$ has a **'Schwarzian' effective action:**

$$S_{\text{Schw.}} = -C \int d\tau \, \{f(\tau), \tau\} \qquad (C \propto N/\mathcal{J})$$

- This action also describes the boundary degree of freedom associated with **dilaton gravity in AdS_2**

[Almheiri/Polchinski '14]
[Maldacena/Stanford/Yang '16]

- The Schwarzian mode describes a **universal & enhanced contribution** to the OTOC:

$$\text{OTOC} = \langle \psi_i(t)\psi_j(0)|\psi_i(t)\psi_j(0)\rangle \sim a_0 - \frac{a_1}{N}\, e^{\frac{2\pi}{\beta}\, t}$$

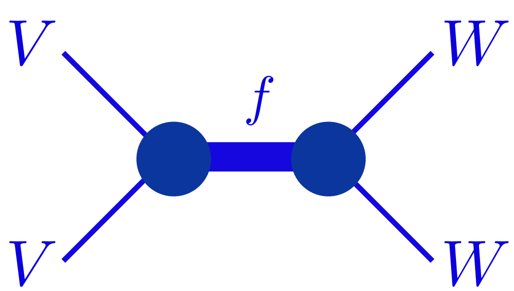
Schwarzian contribution to the OTOC

- Reparametrizations $\tau \to f(\tau)$ couple universally to operators:

$$\langle V(\tau_1)V(\tau_2)\rangle = \frac{1}{\tau_{12}^2} \quad\longrightarrow\quad \left[\frac{f'(\tau_1)f'(\tau_2)}{(f(\tau_1)-f(\tau_2))^2}\right]^{\Delta} \equiv \mathcal{B}_{\Delta}^{(f)}(\tau_1,\tau_2)$$

- This coupling gives universal contribution to 4-point functions:

$$\langle V(\tau_1)V(\tau_2)W(\tau_3)W(\tau_4)\rangle = \int \mathcal{D}f\; e^{-S_{\rm Schw.}[f]}\, \mathcal{B}_{\Delta_V}^{(f)}(\tau_1,\tau_2)\,\mathcal{B}_{\Delta_W}^{(f)}(\tau_3,\tau_4) \;+\; \text{other}$$



- Linearized:

$$f(\tau) = \tau + \epsilon(\tau) \quad\to\quad S_{\rm Schw.} = -\frac{C}{2}\int d\tau\, \epsilon\,(\partial_\tau^2 + \partial_\tau^4)\epsilon + \dots$$

$$\langle V(\tau_1)V(\tau_2)W(\tau_3)W(\tau_4)\rangle = \langle \mathcal{B}_{\Delta_V}^{(\epsilon)}(\tau_1,\tau_2)\,\mathcal{B}_{\Delta_W}^{(\epsilon)}(\tau_3,\tau_4)\rangle + \text{higher order} + \text{other}$$

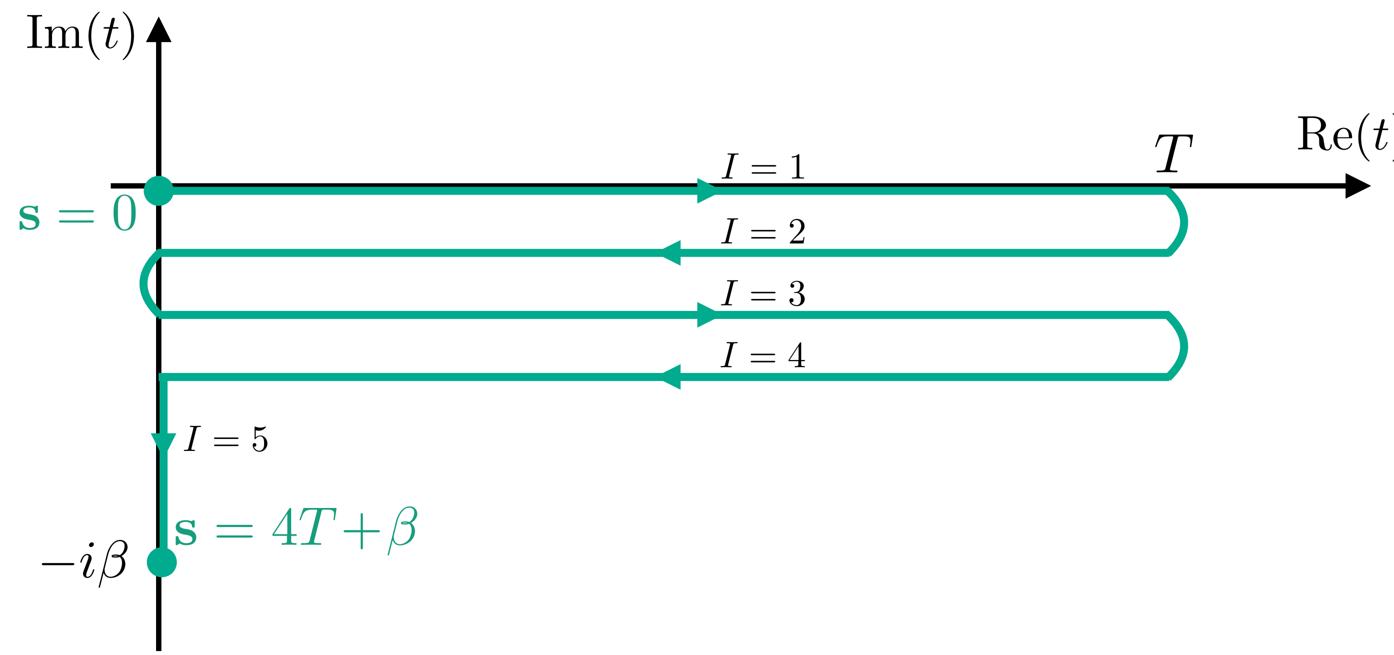
Analytic continuation to OTOC => $\sim -\frac{1}{N} e^{\frac{2\pi}{\beta}t}$

A Lorentzian approach

- So far: only first order in 1/N
- Also cheated a bit: didn't do an actual real-time calculation
- Consider Schwarzian action on OTO contour:

$$S_{\text{Schw.}} = -C \int_{\text{contour}} dt \, \{f(t), t\}$$

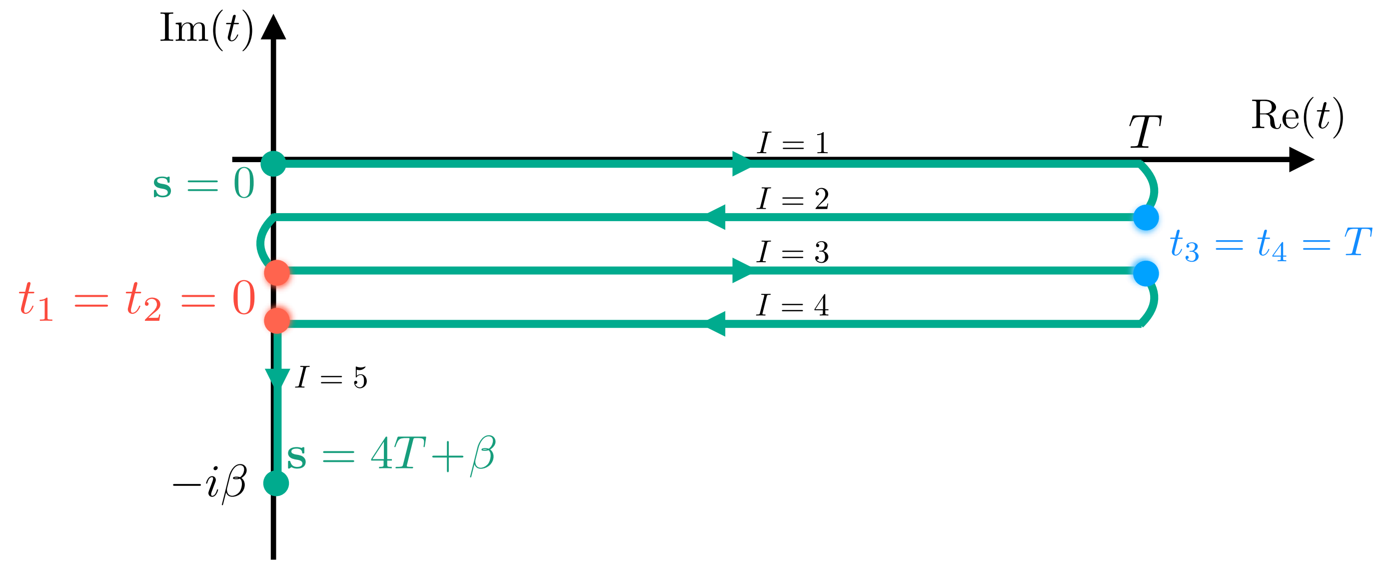
$$(t \equiv t(s))$$



- Among the fluctuations around the thermal saddle $f(t) = \tanh(t/2)$, there are certain **soft modes.**
  - They live on the OTO contour.
  - They can mess up the large-N expansion for large times.

$$\mathcal{F}(t_1,t_2;t_3,t_4) = \int \mathcal{D}f \; e^{iS_{\mathrm{Schw.}}[f]} \, \mathcal{B}^{(f)}_{\Delta_V}(t_1,t_2)\, \mathcal{B}^{(f)}_{\Delta_W}(t_3,t_4)$$

- For large T: path integral dominated by certain ‘scramblon’ modes parametrized by $X^{\pm}$



$$\mathcal{F}(t_1,t_2;t_3,t_4) \approx \int \mathcal{D}X^{+}\mathcal{D}X^{-} \; e^{iS_{\mathrm{eikonal}}[X^{+},X^{-}]} \, \mathcal{B}^{(X^{+})}_{\Delta_V}(t_1,t_2)\, \mathcal{B}^{(X^{-})}_{\Delta_W}(t_3,t_4)$$

$$S_{\mathrm{eikonal}} \propto C\, e^{-T}\, X^{+}X^{-} + \ldots$$

- For $T \gtrsim \log C$: treat integral over $X^{\pm}$ exactly

The 'scramblon' modes

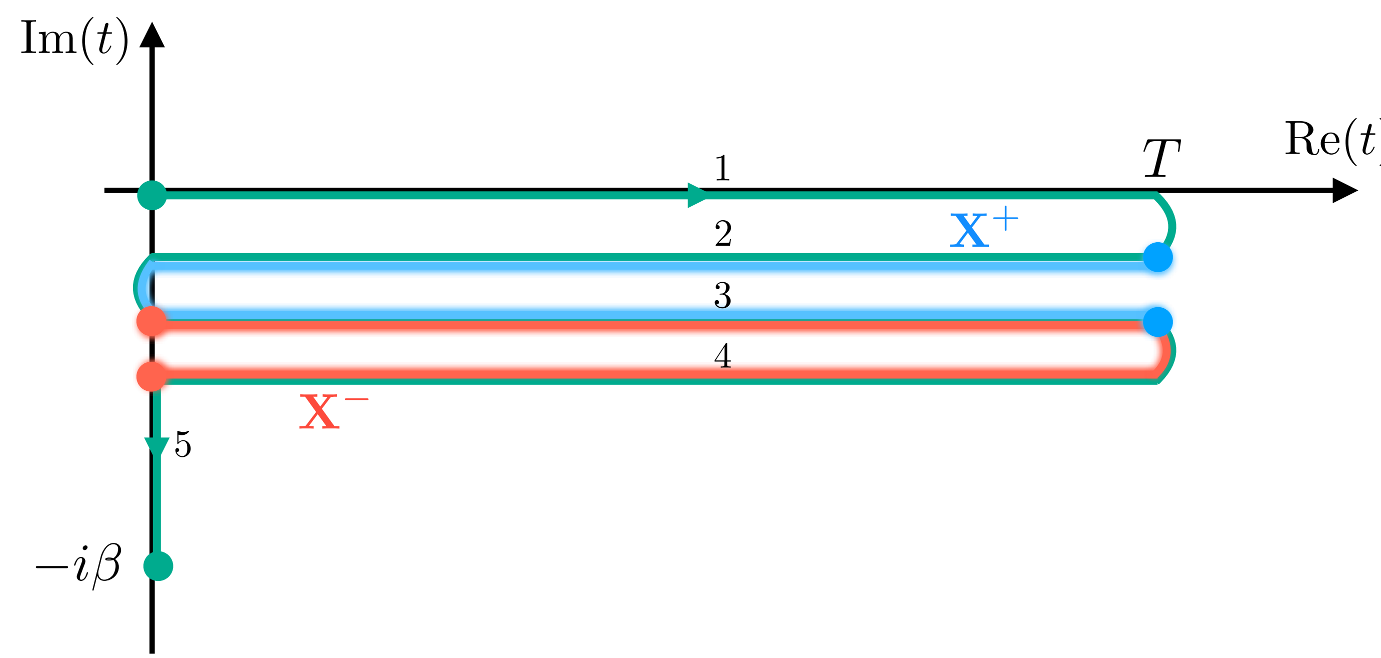
- Zero modes of quadratic action (related to the SL(2,R) symmetry of the Schwarzian):

$$iS_{quad} = \frac{iC}{2}\int_{\text{contour}} dt\, \delta\epsilon_I(\partial_t^4 - \partial_t^2)\delta\epsilon_I$$

soft modes: $\begin{cases} \delta_+\epsilon_I = X^+ e^{-t} \\ \delta_-\epsilon_I = X^- e^{t-T} \end{cases}$

- The modes $\delta_+\epsilon$ $(\delta_-\epsilon)$ can be excited exponentially softly but grow large towards the past (future).

- In the OTOC, these modes interact —> butterfly effect



- Path integral computation of the OTOC:
  - $\delta_+\epsilon_I$ ($\delta_-\epsilon_I$) gets excited by operators in the future (past)

$$\delta\epsilon_I^{\text{otoc}}(s) = (\chi_{I2}+\chi_{I3})\,\delta_+\epsilon_I + (\chi_{I3}+\chi_{I4})\,\delta_-\epsilon_I$$

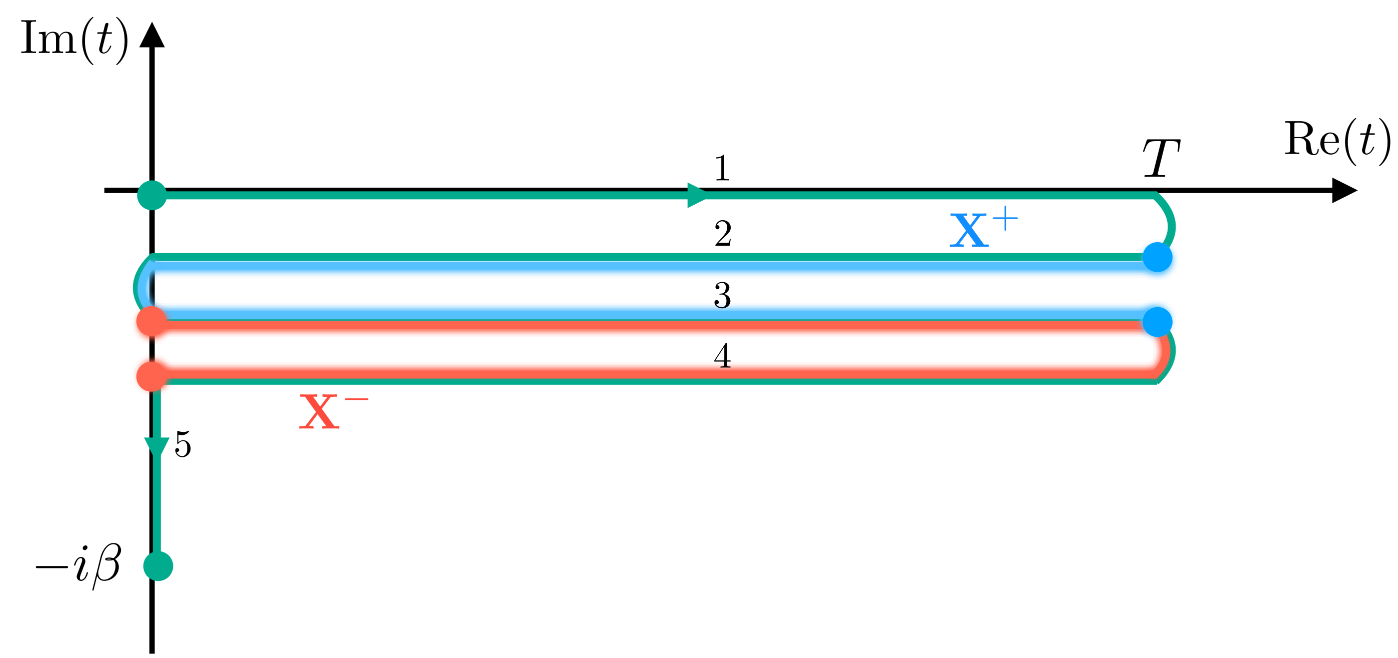
  - Extend to finite SL(2,R) transformations:

$$f_I^{\text{otoc}}(s) = f_I(s) + (\chi_{I2}+\chi_{I3})\,\frac{(1-f_I(s))^2 X^+}{2+(1-f_I(s))X^+} - (\chi_{I3}+\chi_{I4})\,\frac{(1+f_I(s))^2\, e^{-T} X^-}{2+(1+f_I(s))\,e^{-T}X^-}$$

$$f_I(s) = \tanh\frac{t(s)}{2}$$

  - Interaction causes saddle point approximation to break down:

$$S_{\text{quad.}} = \underbrace{2C\, e^{-T}}_{\sim\mathcal{O}(1)}\, X^+ X^-$$



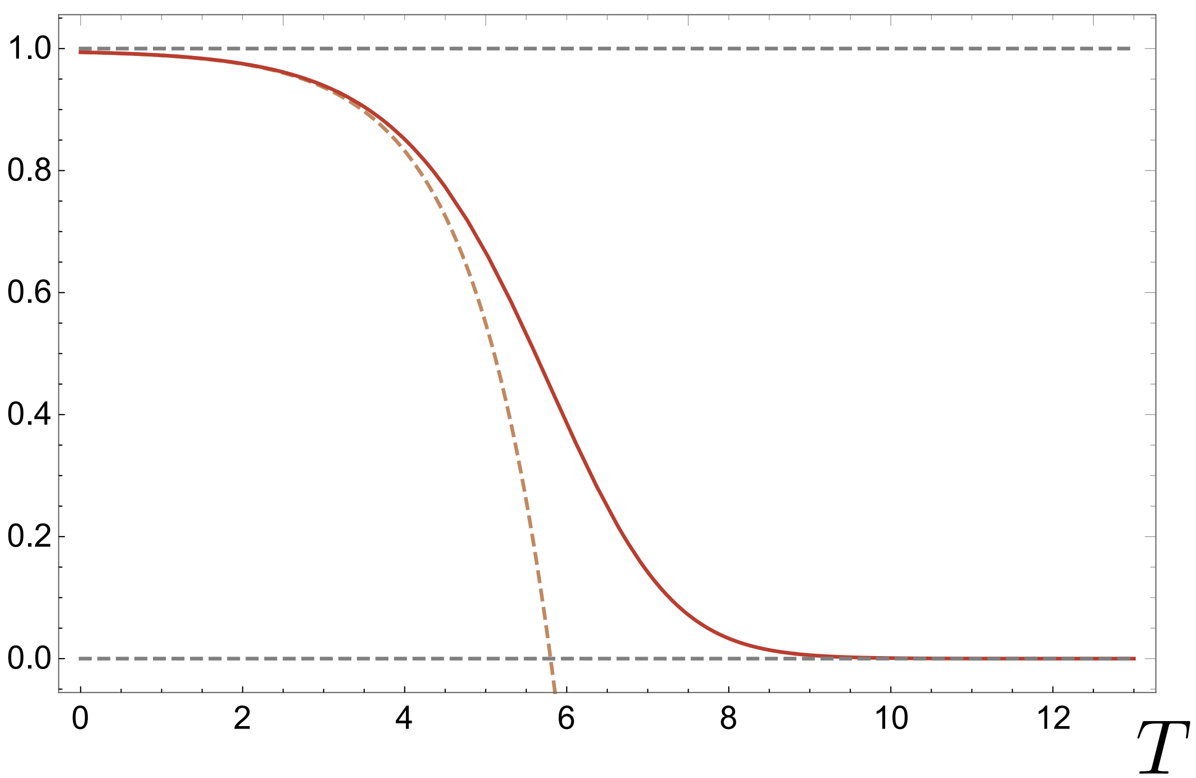
c.f. [Maldacena/Stanford/Yang '16]

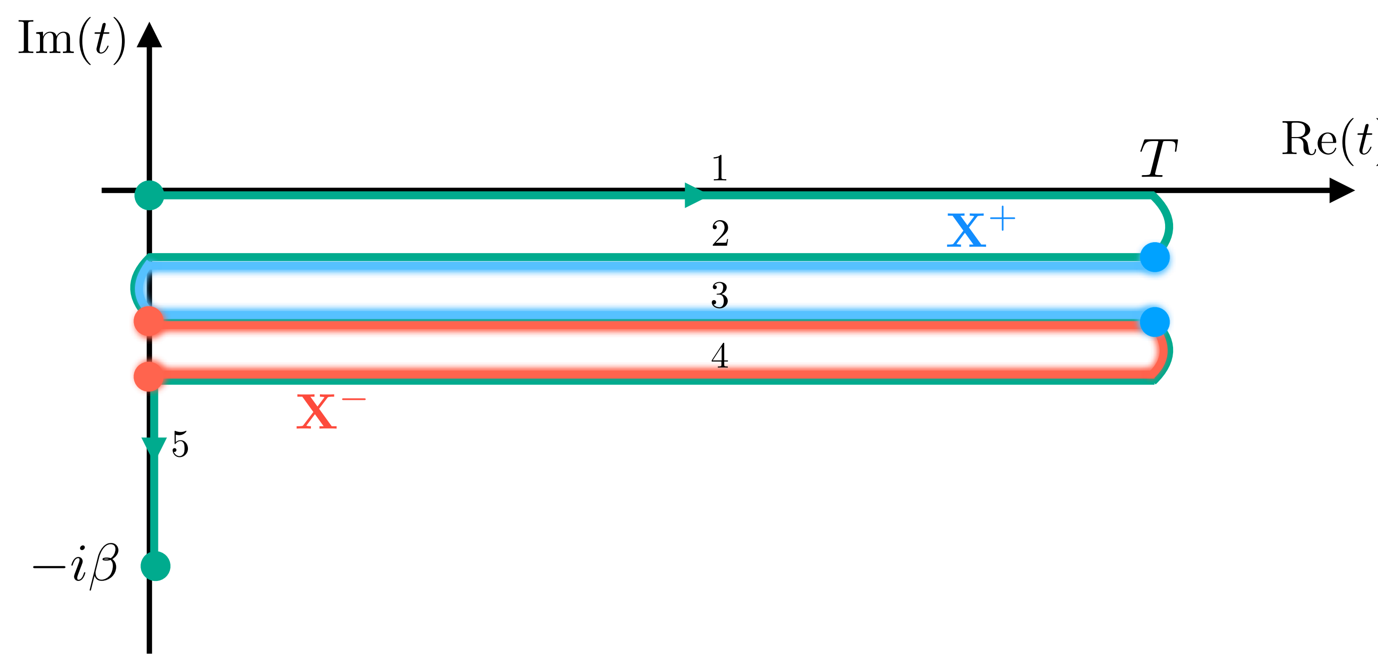
- Soft modes also couple to matter in the same universal way as before: via (finite) SL(2,R) reparametrizations

$$\mathrm{Tr}\left\{W(0)V(T)W(0)V(T)\,e^{-\beta H}\right\} = \int \mathcal{D}X^+\mathcal{D}X^-\; e^{2iC\,e^{-T}X^+X^-}\,\mathcal{B}^{(f^{\rm otoc})}_{\Delta_V}(3T,T)\,\mathcal{B}^{(f^{\rm otoc})}_{\Delta_W}(4T,2T)$$

$$= \langle VV\rangle\langle WW\rangle \times z^{-2\Delta}\,U(2\Delta,1,z^{-1}) \qquad z \propto \frac{e^T}{C}$$

- Bilocal $\mathcal{B}^{(f^{\rm otoc})}_{\Delta_V}$ = two-point function fully backreacted in the presence of a source for X^-





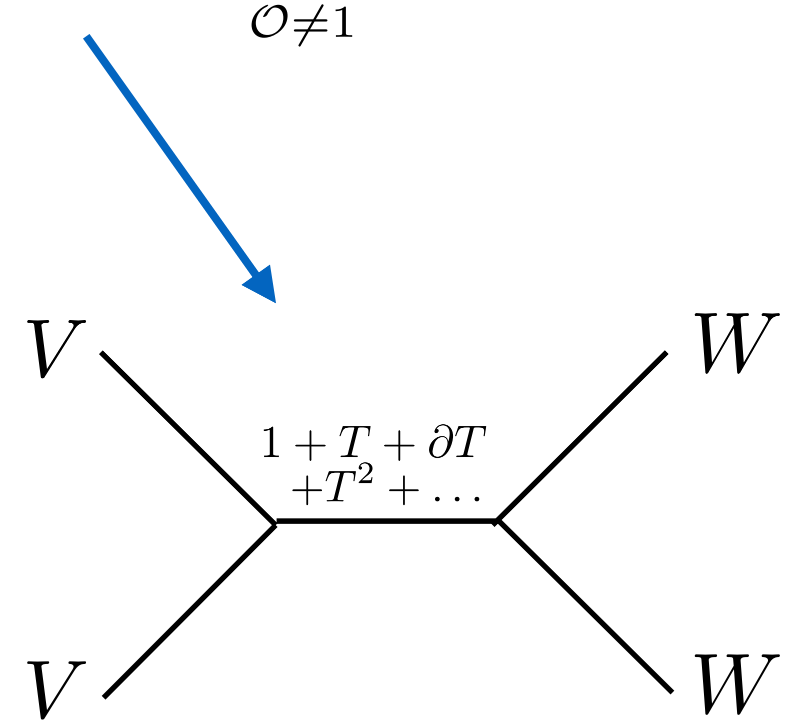
Generalization to CFTs

- The Virasoro identity conformal block plays a similar role in 2d CFTs as the Schwarzian mode exchanges

$$\langle V(z_1)V(z_2)W(z_3)W(z_4)\rangle = \frac{1}{x_{12}^{\Delta_V}\, x_{34}^{\Delta_W}} \left[\mathcal{V}_1\,(z,\bar{z}) + \sum_{\mathcal{O}\neq 1} C_{\mathcal{O}VV} C_{\mathcal{O}WW}\, \mathcal{V}_{\mathcal{O}}(z,\bar{z}) \right]$$

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

- Graviton excitations in AdS



- Contribution to the OTOC matches the black hole result: fast scrambling, maximal Lyapunov exponent

[Roberts/Stanford '14] ... [FH/Reeves/Rozali '19]

Generalization to CFTs

- 2d reparametrizations $(z,\bar{z}) \to (f(z),\bar{f}(\bar{z}))$ are broken by the vacuum to *SL(2,R) x SL(2,R)*
- They acquire an action, which realizes this structure non-linearly [Polyakov '87] [Alekseev/Shatashvili '89] [Cotler/Jensen '18]
- Shortcut: consider generating functional for *T*-correlators:

$$(z,\bar{z}) \to (z+\epsilon, \bar{z}+\bar{\epsilon}) \qquad S_{CFT} \longrightarrow S_{CFT} + \int d^2z\ \left\{\bar{\partial}\epsilon\, T(z) + \partial\bar{\epsilon}\, \bar{T}(\bar{z})\right\}$$

$$e^{-W[\epsilon]+\int d^2z\,\bar{\partial}\epsilon\, T_\epsilon(z)} = \frac{1}{Z_0}\int [d\Phi]\, e^{-S_{CFT}-\int d^2z\,\bar{\partial}\epsilon\, T(z)}$$

quadratic action: $W_2[\epsilon] = \frac{c}{24\pi}\int dt d\sigma \left[\bar{\partial}\epsilon\,(\partial_t^3 - \partial_t)\epsilon + \text{anti-holo.}\right]$

[FH/Rozali '18] [FH/Reeves/Rozali '19] [Nguyen '21]

Generalization to CFTs

$$W_2[\epsilon] = \frac{c}{24\pi}\int dt d\sigma \left[\bar{\partial}\epsilon\,(\partial_t^3 - \partial_t)\epsilon + \text{anti-holo.}\right]$$

- Zero modes $X^+(\sigma)e^{-t}$ and $X^-(\sigma)e^{t-T}$

  => eikonal action, OTOC etc. as before

$$\text{OTOC} = z^{-2h'}U\left(2h', 1+2h'-2h, \frac{1}{z}\right) \qquad z \equiv -\frac{12\pi i}{c}\,\Theta(\sigma'-\sigma)\,e^{T+(\sigma-\sigma')}$$

c.f. [Chen/Fitzpatrick/Kaplan/Li/Wang '16]

- Mechanism quite general for theories with SL(2,R) symmetry
- EFT of maximal chaos based on symmetry [Blake/Lee/Liu '18]
- **What about less constrained cases, sub-maximal chaos?**

Sub-maximal chaos in SYK at large q

[Choi/FH/Mezei/Sarosi 2301.05698]

Large q SYK model

- Consider SYK with

$$H = i^{\frac{q}{2}} \sum_{1\le i_1<\ldots<i_q\le N} j_{i_1\cdots i_q}\, \psi_{i_1}\cdots\psi_{i_q} \qquad \langle j^2_{i_1\cdots i_q}\rangle = \frac{2^{q-1}\mathcal{J}^2(q-1)!}{qN^{q-1}}$$

  - Integrate out fermions and disorder (using self-averaging property)
  - Obtain non-local effective action in terms of G, Σ
  - **Large q limit:** $N \gg q^2 \gg 1\,,\ \ \mathcal{J}^2 = \text{fix}$
  - Leading order: bi-local 'Liouville' action:

$$S[g] = \frac{N}{4q^2}\int d\tau_1 d\tau_2 \left[\frac{1}{4}\partial_{\tau_1} g\,\partial_{\tau_2} g - \mathcal{J}^2\, e^g\right]$$

$$G(\tau_1,\tau_2) = G_{\text{free}}(\tau_{12})\left[1+\frac{1}{q}\,g(\tau_1,\tau_2)+\mathcal{O}(q^{-2})\right]$$

[Maldacena/Stanford '16] [Cotler et al. '17]

- The leading connected OTOC is known at any coupling:

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\langle T_{\mathcal{C}}\{\psi_i(t_1)\psi_i(t_2)\psi_j(t_3)\psi_j(t_4)\}\rangle_\beta \equiv G(t_{12})G(t_{34}) + \frac{1}{N}\mathcal{F}(t_1,t_2,t_3,t_4) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

[Streicher '19]
[Choi/Mezei/Sarosi '19]

$$\mathcal{F}^{\rm OTOC}(0,0,T,T) = -2\sec\left(\frac{v\pi}{2}\right)^3\cosh\left(\frac{v}{2}(i\pi-2T)\right) + \frac{2\tan\left(\frac{\pi v}{2}\right)}{\frac{\pi v}{2}+\cot\left(\frac{\pi v}{2}\right)}\left(1+\frac{\pi v}{2}\tan\left(\frac{\pi v}{2}\right)\right)^2 + \tan\left(\frac{\pi v}{2}\right)^3 iv(i\pi-2T)$$

$$= \alpha\, e^{vT} + \mathcal{O}\big((e^{vT})^0\big)$$

$$\beta\mathcal{J} = \pi v \sec\left(\frac{\pi v}{2}\right)$$

$$0 \le v \le 1$$

- How to get this from a ‘scramblon' EFT?

- Extend to higher orders in 1/N

- [Gu/Kitaev/Zhang '21] : theories dominated by ladder diagrams (including large-q SYK) have eikonal action giving sub-maximal chaos

- We will instead use path integral approach, developed for a simpler model ('Brownian SYK') in [Stanford/Yang/Yao '21] .

- Place the bi-local Liouville theory on the OTO contour:

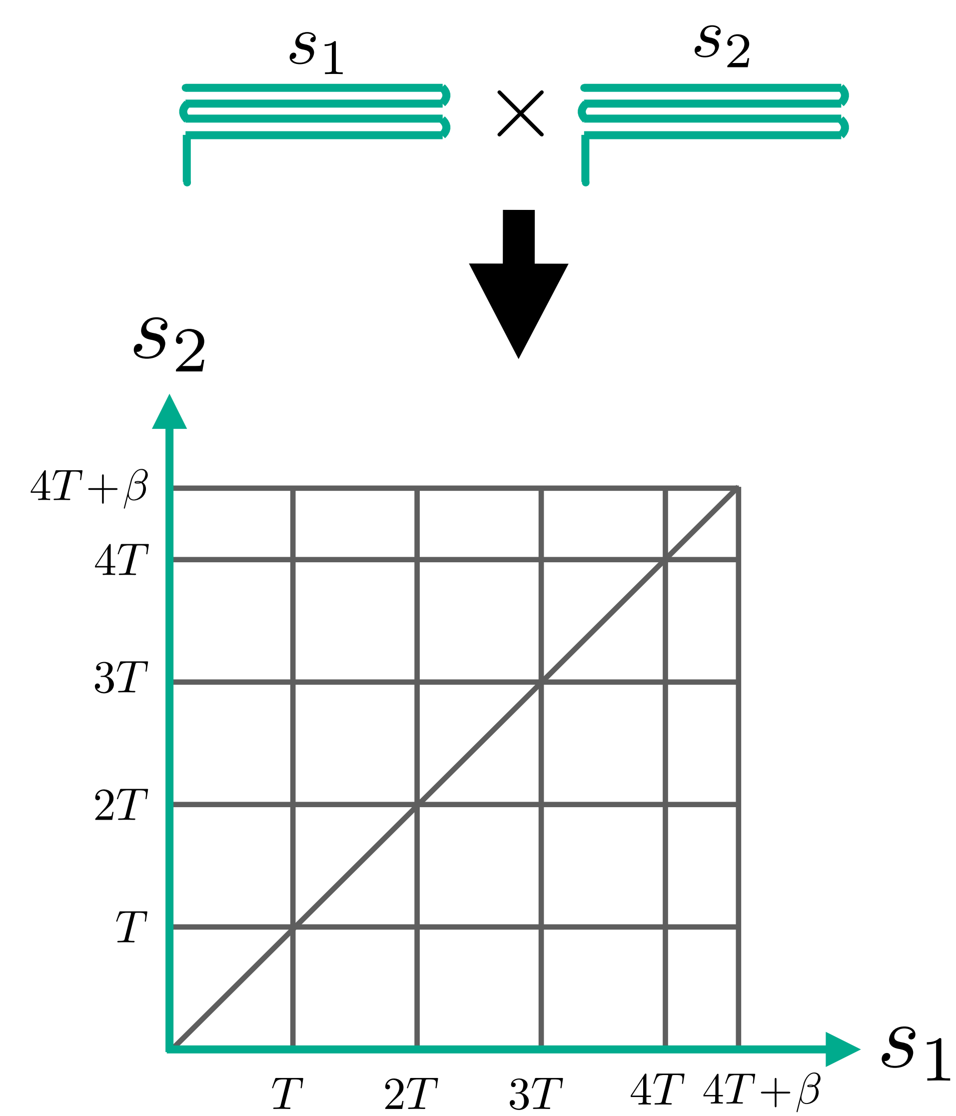
$$\frac{1}{2}\partial_1\partial_2 g_{IJ}(s_1,s_2) \pm \mathcal{J}^2\, e^{g_{IJ}(s_1,s_2)} = 0$$

  - General solution:

$$e^{g_{IJ}(s_1,s_2)} = \pm\frac{1}{\mathcal{J}^2}\frac{F'_{IJ}(s_1)G'_{IJ}(s_2)}{(F_{IJ}(s_1)-G_{IJ}(s_2))^2}$$

  - $\mathrm{diff}(S^1)\times\mathrm{diff}(S^1)\to SL(2,R)_{\mathrm{diag}}$:

$$F_{IJ}(s) \longrightarrow \frac{a\,F_{IJ}(s)+b}{c\,F_{IJ}(s)+d}\,, \qquad G_{IJ}(s) \longrightarrow \frac{a\,G_{IJ}(s)+b}{c\,G_{IJ}(s)+d}$$



- Saddle: $F_{IJ}(s) = \tanh\left(\frac{vt(s)+2\pi vi}{2} + c_{IJ}\right)$, $G_{IJ}(s) = \tanh\left(\frac{vt(s)+\pi(1+v)i}{2} + c_{IJ}\right)$

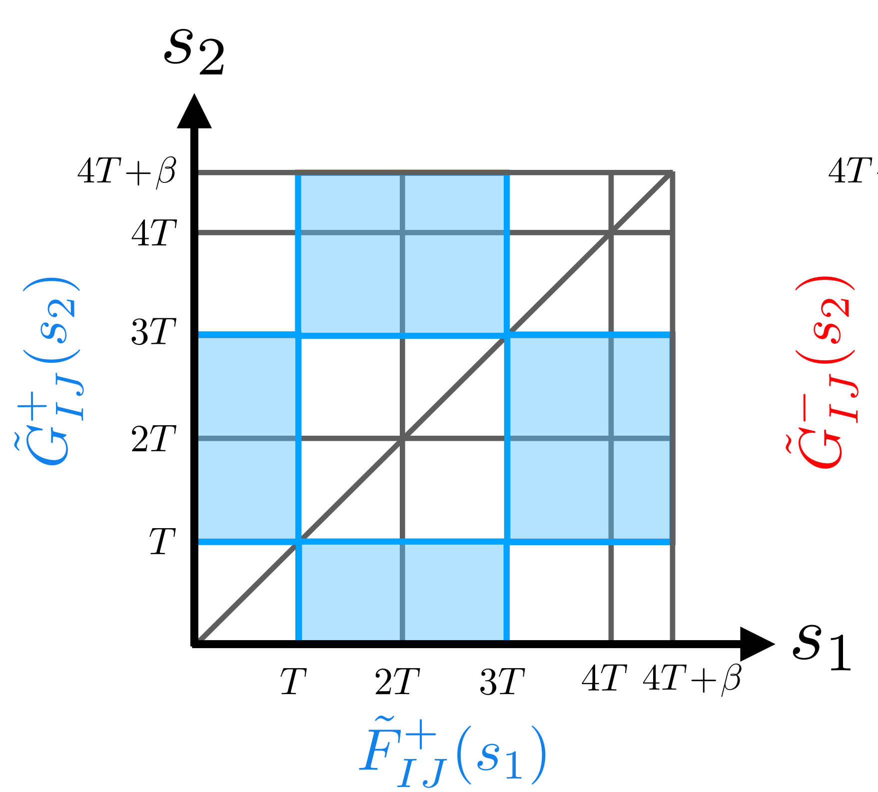
- Again, the bilocal quadratic action of fluctuations has nearly-zero modes. They are generated by $SL(2,R)_{\text{diag}}$:

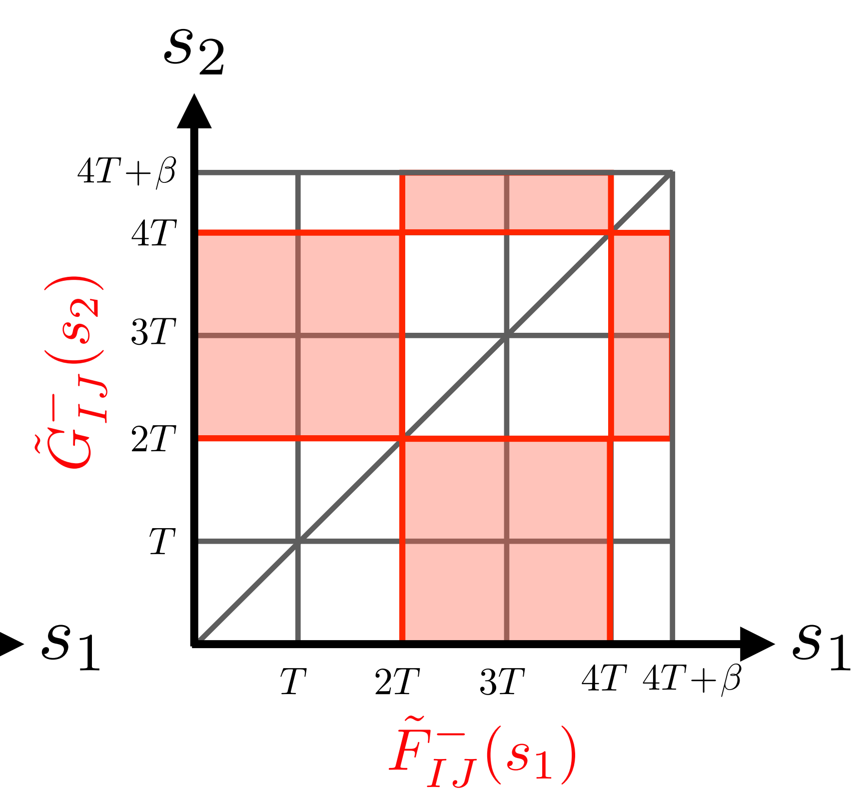
$$F^{\text{otoc}}_{IJ}(s_1) = F_{IJ} + (\delta_{I2}+\delta_{I3})\tilde{F}^{+}_{IJ} + (\delta_{I3}+\delta_{I4})\tilde{F}^{-}_{IJ}$$

$$G^{\text{otoc}}_{IJ}(s_2) = G_{IJ} + (\delta_{I2}+\delta_{I3})\tilde{G}^{+}_{IJ} + (\delta_{I3}+\delta_{I4})\tilde{G}^{-}_{IJ}$$

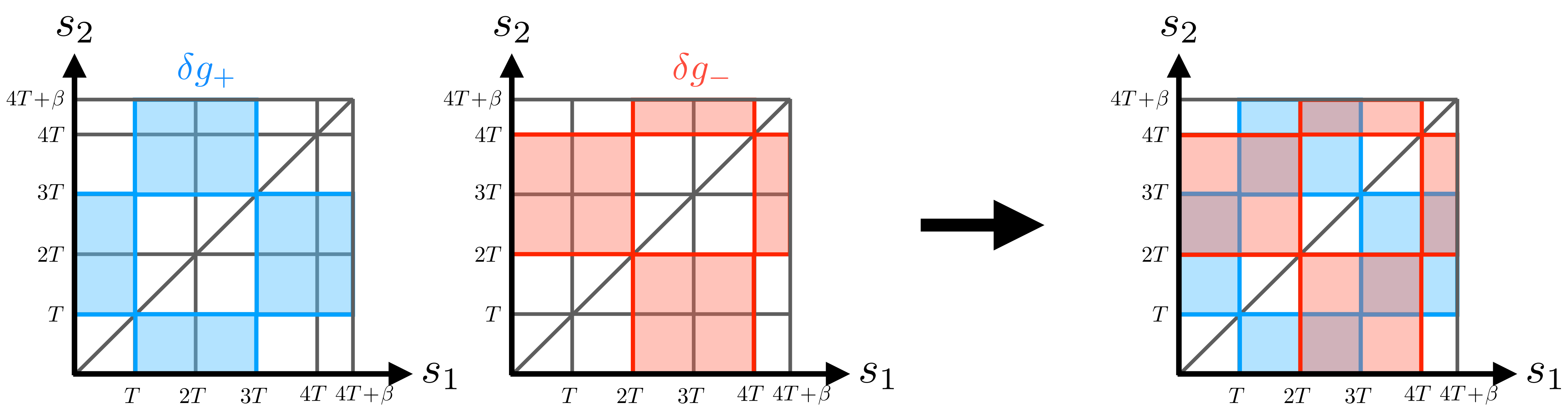
$$\tilde{F}^{+}_{IJ} = \frac{X^+(1-F_{IJ})^2}{2+X^+(1-F_{IJ})} \quad \text{etc.}$$

- Can fix c_{IJ} such that fluctuations satisfy KMS (!)





- The eikonal action is obtained by evaluating $S_{\text{quad.}}[\delta_+ g, \delta_- g]$



$$S_{\text{eikonal}} = \frac{N}{2q^2} \cos\left(\frac{\pi v}{2}\right) e^{\frac{i\pi v}{2} - vT} X^+ X^-$$

$$\text{OTOC} = z^{-2\Delta} U(2\Delta, 1, z^{-1}) \qquad \left(z = \frac{1}{4\Delta^2 N} \sec\left(\frac{\pi v}{2}\right)^3 e^{-\frac{i\pi v}{2} + vT}\right)$$

- Effective action for chaos with $\lambda_L = \dfrac{2\pi v}{\beta} \quad (0 < v < 1)$

[Choi/FH/Mezei/Sarosi '23]

see also: [Gao/Liu '23]

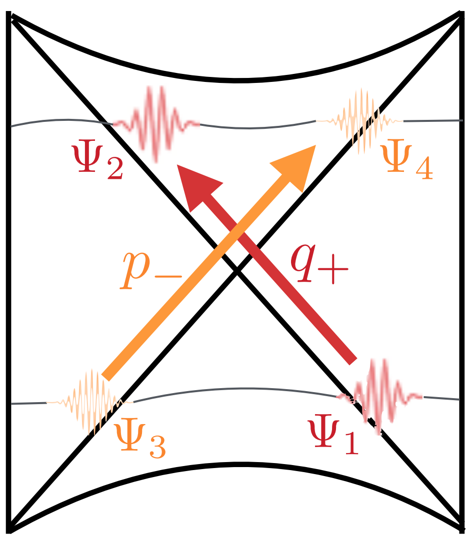
Comparison with stringy effects

Wave function representation: Schwarzian

$$\mathrm{OTOC} \approx \int \mathcal{D}X^+ \mathcal{D}X^- \; e^{iS_{\mathrm{eikonal}}[X^+,X^-]} \, \mathcal{B}^{(X^+)}_{\Delta_V}(t_1,t_2) \, \mathcal{B}^{(X^-)}_{\Delta_W}(t_3,t_4)$$

$$S_{\mathrm{eikonal}} = 2C\, e^{-T} X^+ X^-$$

- Write vertex factors using 'bulk' wave functions:



bulk-boundary propagators: $\Psi_j \equiv x_j^{\Delta} e^{-x_j}$

$$x_1 = -iq_+ e^{t_1}\,, \quad x_2 = iq_+ e^{t_2}$$
$$x_3 = ip_- e^{T-t_3}\,, \quad x_4 = -ip_- e^{T-t_4}$$

$$\mathcal{B}^{(X^+)}_{\Delta_V}(t_1,t_2) \propto \int dq_+ \left[\frac{\Psi_1(q_+)\Psi_2(q_+)}{-q_+}\right] e^{-iq_+X^+}$$

$$\mathcal{B}^{(X^-)}_{\Delta_W}(t_3,t_4) \propto \int dp_- \left[\frac{\Psi_3(p_-)\Psi_4(p_-)}{-p_-}\right] e^{-ip_-X^-}$$

$$\mathrm{OTOC} \propto \int_{-\infty}^{0} dq_+ dp_- \left[\frac{\Psi_1(q_+)\Psi_2(q_+)}{-q_+}\right]\left[\frac{\Psi_3(p_-)\Psi_4(p_-)}{-p_-}\right] e^{i\delta}\,, \qquad \delta = -\frac{e^T}{2C}\, q_+ p_-$$

eikonal phase

Wave function representation: large q

- Similarly, for large q SYK:

$$\text{OTOC} \propto \int_{-\infty}^{0} d\mathfrak{q}_+ d\mathfrak{p}_- \left[\frac{\Psi_1^s(\mathfrak{q}_+)\Psi_2^s(\mathfrak{q}_+)}{-\mathfrak{q}_+}\right]\left[\frac{\Psi_3^s(\mathfrak{p}_-)\Psi_4^s(\mathfrak{p}_-)}{-\mathfrak{p}_-}\right] e^{i\delta}\,, \qquad \delta = -i\frac{2q^2}{N\cos\left(\frac{\pi v}{2}\right)}\left(-ie^T\mathfrak{q}_+\mathfrak{p}_-\right)^v$$

$$\Psi_j^s \equiv x_j^{v\Delta}\, e^{-x_j^v}$$

Compare: Einstein gravity + stringy corrections:

$$\delta \sim G_N \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{e^{ipx}}{p^2+\mu^2} \left(-ic_0\,\alpha' e^T \mathfrak{p}_+\mathfrak{q}_+\right)^{1-\alpha'(p^2+\mu^2)/2r_0^2}$$

[Stanford/Shenker '14] [Nezami et al. '21]

- No momentum integral in SYK. Otherwise very similar structure!

Large q SYK chain

Large q SYK chain

- Similar system with spatial dimension: chain of SYK models

$$H = i^{\frac{q}{2}} \sum_{x=0}^{M-1} \left[\sum_{1\leq i_1 < \ldots < i_q \leq N} j_{i_1 \cdots i_q} \psi_{i_1} \cdots \psi_{i_q} + \sum_{\substack{i_1,\ldots,i_{q/2} \\ j_1,\ldots,j_{q/2}}} j'_{i_1 \cdots i_{q/2} j_1 \cdots j_{q/2}} \psi_{i_1} \cdots \psi_{i_{q/2}} \psi_{j_1} \cdots \psi_{j_{q/2}} \right]$$

[Gu/Qi/Stanford '16]

- Large q limit:

$$S = \frac{N}{4q^2} \sum_{x=0}^{M-1} \int ds_1 ds_2 \left[\frac{1}{4} \partial_1 g_x \partial_2 g_x - \mathcal{J}_0^2 r_{s_1} r_{s_2} e^{g_x} - \mathcal{J}_1^2 r_{s_1} r_{s_2} e^{\frac{1}{2}(g_x + g_{x+1})} \right]$$

[Choi/Mezei/Sarosi '20]

- Saddle point solution is spatially homogeneous:

$$e^{g^*_{IJ}(s_1,s_2)} = \frac{1}{\mathcal{J}^2 \, r_{s_1} r_{s_2}} \frac{F'_{IJ}(s_1) G'_{IJ}(s_2)}{(F_{IJ}(s_1) - G_{IJ}(s_2))^2}, \quad s_1 \in I, s_2 \in J$$

- Follow same procedure as before (OTO contour, zero modes…)
- Find momentum-dependent scramblon action:

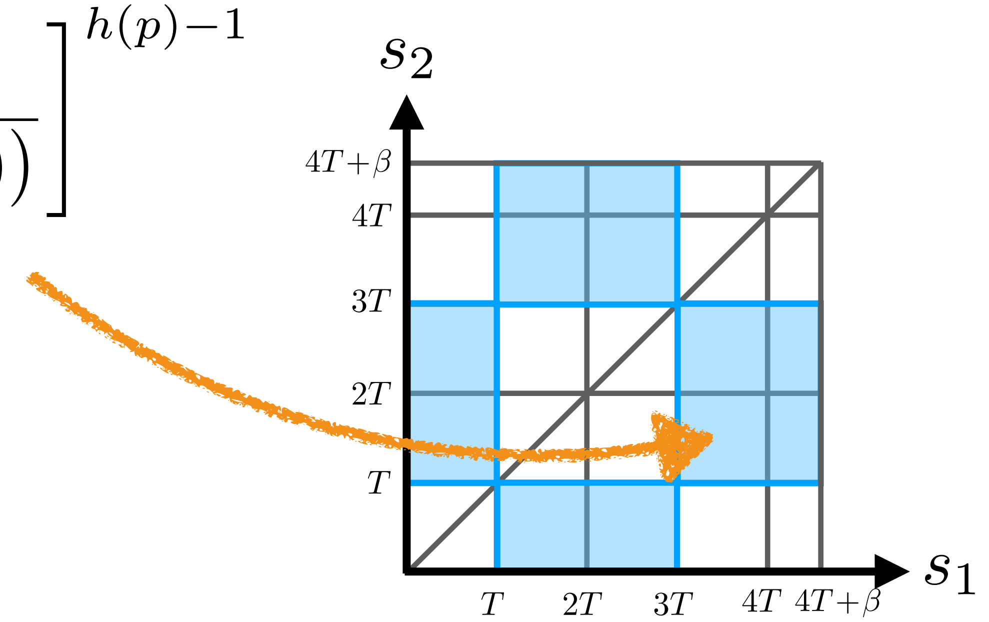
$$iS_{eik} = \int_{-\pi}^{\pi} \frac{dp}{2\pi}\, iC_{eik}(p)\, X^+(p)X^-(-p)\,,$$

$$iC_{eik}(p) = -\frac{N}{2q^2}\frac{e^{\left(\frac{i\pi}{2}-T\right)\kappa}}{\sqrt{\pi}}\cos\left(\frac{\pi\kappa}{2}\right)\cos\left(\frac{\pi\kappa}{v}\right)\Gamma\left(\frac{\kappa}{v}+1\right)\Gamma\left(\frac{1}{2}-\frac{\kappa}{v}\right)$$

$$\kappa = v(h-1)\,, \qquad \frac{h(h-1)}{2} = 1 + \frac{\gamma}{2}[\cos p - 1]$$

- Second required ingredient: vertex functions $e^{\Delta\, g^{\pm}_{IJ,x}(t_1,t_2)}$
  - To linear order in $X^{\pm}(k)$: easy, e.g.

$$\delta_+ g_{42,p}(t_1,t_2) = \left[e^{-\frac{3\pi i v}{2}}\frac{e^{-v(t_1+t_2)/2}}{\cos\left(\frac{v}{2}(\pi - it_{12})\right)}\right]^{h(p)-1}$$



- OTOC to leading order in 1/N:

$$\mathrm{OTOC}\big|_{\mathcal{O}(1/N)} \sim \frac{1}{N}\int \frac{dp}{2\pi}\,\frac{e^{ipx+v(h(p)-1)\left(T-\frac{i\pi}{2}\right)}}{d(p)} \qquad h(p=0)=2$$

  - Small x/T : saddle point dominates

$$\sim \frac{c(v,\gamma)}{N}\times e^{v\left(-\frac{i\pi}{2}+T\right)}\times \frac{e^{-\frac{x^2}{2r^2}}}{\sqrt{2\pi r^2}}\,, \qquad r^2=\frac{\gamma v}{3}\left(-\frac{i\pi}{2}+T\right)$$

  - Large x/T : pole $d(k)=0$ dominates

$$\sim \frac{c'(v,\gamma)}{N}\times e^{T-x/u_B^{(T)}} \qquad u_B^{(T)}=\left[\operatorname{arccosh}\left(\frac{1+v+(\gamma-2)v^2}{\gamma v^2}\right)\right]^{-1}$$

  - Mechanism as in stringy corrections to gravity eikonal phase!

    [Shenker/Stanford ’14] [Mezei/Sarosi ’19] [FH/Choi/Mezei/Sarosi ’23]

Large q chain: higher orders

- Vertex functions are no longer generated by any symmetry. We find:

complicated!
only way to simplify
them is for small *p*
or small *(v-1)*

momentum-dependent
coefficients

homogeneous saddle

$$e^{\Delta\, g^{+}_{42,x}(t_1,t_2)} = e^{\Delta\, g_{42}(t_1,t_2)} \Bigg\{ 1 + \Delta \sum_{n\geq 1} \int \frac{dp_1 \cdots dp_n}{(2\pi)^n} \frac{\mathfrak{b}_{\Delta,p_1,\ldots,p_n}}{\mathfrak{b}_{\Delta,p_1} \cdots \mathfrak{b}_{\Delta,p_n}} A_n(p_1,\ldots,p_n)$$
$$\times \left[e^{-\frac{3i\pi v}{2}} \frac{e^{-v(t_1+t_2)/2}}{\cos\left(\frac{v}{2}(\pi - it_{12})\right)} \right]^{h_1+\ldots+h_n-n} X_1^+ \cdots X_n^+ \, e^{i(p_1+\ldots+p_n)x} \Bigg\}$$

zero mode of SYK ‘dot’
raised to a power $\in [0,n]$

Large q chain: higher orders

- Vertex functions are no longer generated by any symmetry. We find:

$$e^{\Delta\, g^+_{42,x}(t_1,t_2)} = e^{\Delta\, g_{42}(t_1,t_2)} \Bigg\{ 1 + \Delta \sum_{n\geq 1} \int \frac{dp_1 \cdots dp_n}{(2\pi)^n} \frac{\mathfrak{b}_{\Delta,p_1,\ldots,p_n}}{\mathfrak{b}_{\Delta,p_1}\cdots \mathfrak{b}_{\Delta,p_n}} A_n(p_1,\ldots,p_n) \times \left[e^{-\frac{3i\pi v}{2}} \frac{e^{-v(t_1+t_2)/2}}{\cos\left(\frac{v}{2}(\pi - it_{12})\right)} \right]^{h_1+\ldots+h_n-n} X_1^+ \cdots X_n^+ \, e^{i(p_1+\ldots+p_n)x} \Bigg\}$$

$$\mathrm{OTOC}_{eik} \propto \int_{-\infty}^{0} d\mathfrak{q}_+ d\mathfrak{p}_- \left[\frac{\Psi_1^s(\mathfrak{q}_+)\Psi_2^s(\mathfrak{q}_+)}{-\mathfrak{q}_+} \right] \left[\frac{\Psi_3^s(\mathfrak{p}_-)\Psi_4^s(\mathfrak{p}_-)}{-\mathfrak{p}_-} \right] \times \exp\Bigg\{ \int \frac{dp}{2\pi} \frac{1}{\mathfrak{b}^2_{\Delta,p}} \frac{(-i\mathfrak{q}_+\mathfrak{p}_- e^T)^{\kappa(p)}}{i\mathfrak{C}(p)} e^{ip(x-x')} + \int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi} \frac{2}{\mathfrak{b}^2_{\Delta,p_1}\mathfrak{b}^2_{\Delta,p_2}} \left(A_2^2 - \frac{1}{4} \right) \frac{(-i\mathfrak{q}_+\mathfrak{p}_- e^T)^{\kappa(p_1)+\kappa(p_2)}}{i\mathfrak{C}(p_1)\, i\mathfrak{C}(p_2)} e^{i(p_1+p_2)(x-x')} + \ldots \Bigg\}$$

To do: understand the meaning of these terms! (“multi-string effects”?)

Outlook

Some questions

- EFT of chaos in CFTs:
  - d=2n CFTs: quadratic action of reparametrization modes derived from conformal anomalies [FH/Reeves/Rozali '19]. Is there a non-linear and/or eikonal action?
  - Corrections to maximal chaos from conformal Regge theory [Kravchuk/Simmons-Duffin '19]
- **Conceptual questions:** what's the role of symmetries? Do they play any role in the "stringy" mechanism for sub-maximal chaos?
  - Relate to symmetry-based approaches, e.g., [Blake/Lee/Liu '18]
- **Applications:** e.g., large q SYK chain
  - Study spatial propagation of chaos at late times analytically
  - Multi-string effects?