# Spin glasses and holography

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# Motivation

- Disorder and averaging play crucial role in recent AdS/CFT discussions

| | | | |
|---|---|---|---|
| JT gravity | ⟷ | low-energy dynamics of SYK-type ensembles | [Kitaev][Maldacena/Stanford (/Yang)][Engelsoy/Mertens/Verlinde][Maldacena/Qi][Blommaert/Mertens/Verschelde] ... |
| $AdS_3$ pure gravity, 'U(1) gravity' | ⟷ | CFT ensembles, microcanonical averaging | [Maloney/Witten][Afkhami-Jeddi/Cohn/Hartman/Tajdini][Cotler/Jensen][Pollack/Rozali/Sully/Wakeham] ... |
| wormholes, gravitational instantons | ⟷ | spectral form factor, unitary Page curve, statistics of OPE coefficients | [Cotler +8][Saad/Shenker/Stanford][Almheiri/Engelhardt/Marolf/Maxfield][Penington][Almheiri/Hartman/Maldacena/Shaghoulian/Tajdini][Saad][Belin/de Boer] ... |

Desirable to understand other characteristic properties of disordered systems and consequences for gravity!

# Outline

- Introduction
- Random SU(M) Heisenberg model
- Holographic speculations
- p-spin spherical model
- Conclusion

# Introduction

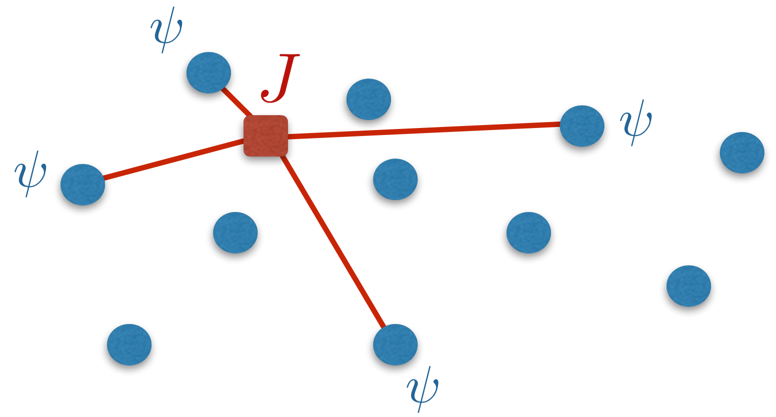
# SYK model: important features

- $N$ Majorana fermions with random, Gaussian couplings

$$H = -\sum_{i,j,k,l=1}^{N} J_{ijkl}\psi_i\psi_j\psi_k\psi_l$$

- Solvable for $N \gg \beta J \gg 1$

$$(\overline{J_{ijkl}} = 0, \quad \overline{J_{ijkl}^2} = J^2/N^3)$$




- 'Mean field' description at large N in terms of bilocal 2-point function

$$G(\tau,\tau') = \frac{1}{N}\sum_{i=1}^{N}\langle\psi_i(\tau)\psi_i(\tau')\rangle$$

- Emergent reparametrization invariance: $\tau \to f(\tau)$
  - Broken by saddle point solution to $f \in SL(2,\mathbb{R})$

$$G_c(\tau-\tau') \propto \frac{1}{(\tau-\tau')^{2/q}}$$

[Sachdev/Ye '93] [Kitaev '15]
[Maldacena/Stanford '16] ...

- The pseudo-Goldstone associated with reparametrizations $\tau \to f(\tau)$ has a **'Schwarzian' effective action:**

$$I_{\text{Schw.}} \propto -\frac{N}{\mathcal{J}} \int d\tau \, \{f(\tau), \tau\}$$

- Same action also describes **dilaton gravity in $AdS_2$**

[Almheiri/Polchinski '14] [Maldacena/Stanford/Yang '16]

- Symmetry breaking pattern implies near-extremal entropy of the form

$$S = S_0 + \# \frac{N}{\beta \mathcal{J}}$$

from Schwarzian

- Schwarzian gives universal leading contribution to **out-of-time-order correlation functions (OTOCs)**

$$\text{OTOC} = \langle \psi_i(t)\psi_j(0) | \psi_i(t)\psi_j(0) \rangle \sim a_0 - \frac{a_1}{N} e^{\left(\frac{2\pi}{\beta} + \ldots\right)t}$$

[Kitaev '15] [Maldacena/Stanford '16] ...

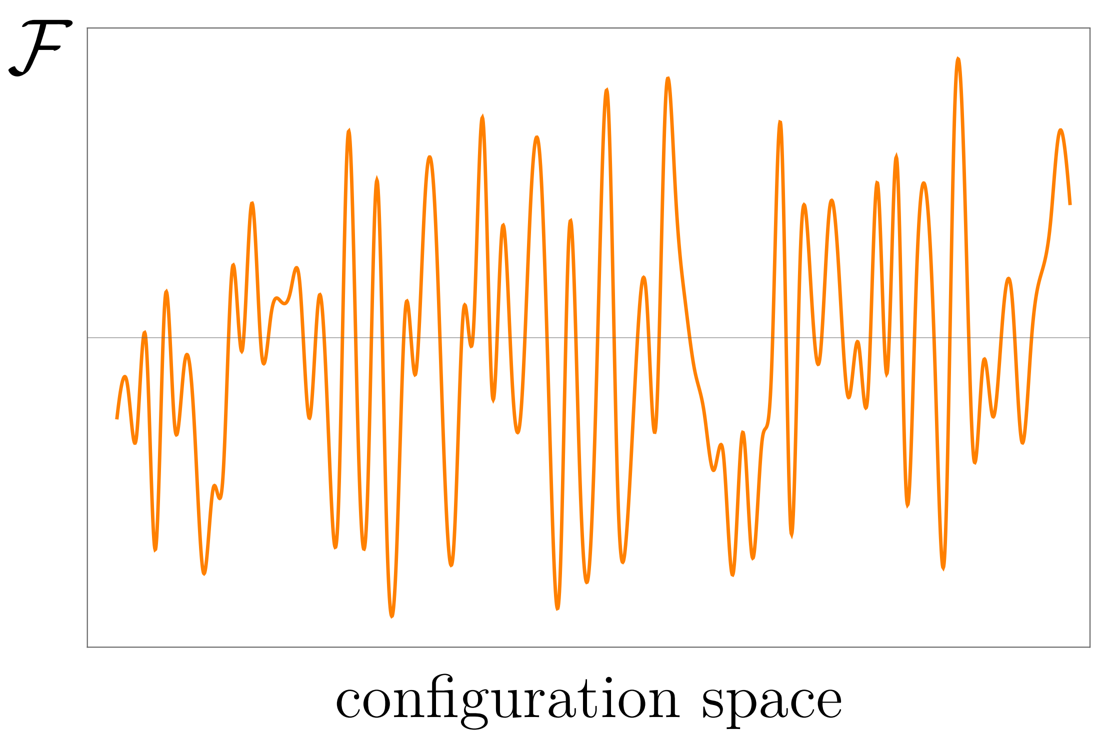
# Spin glasses

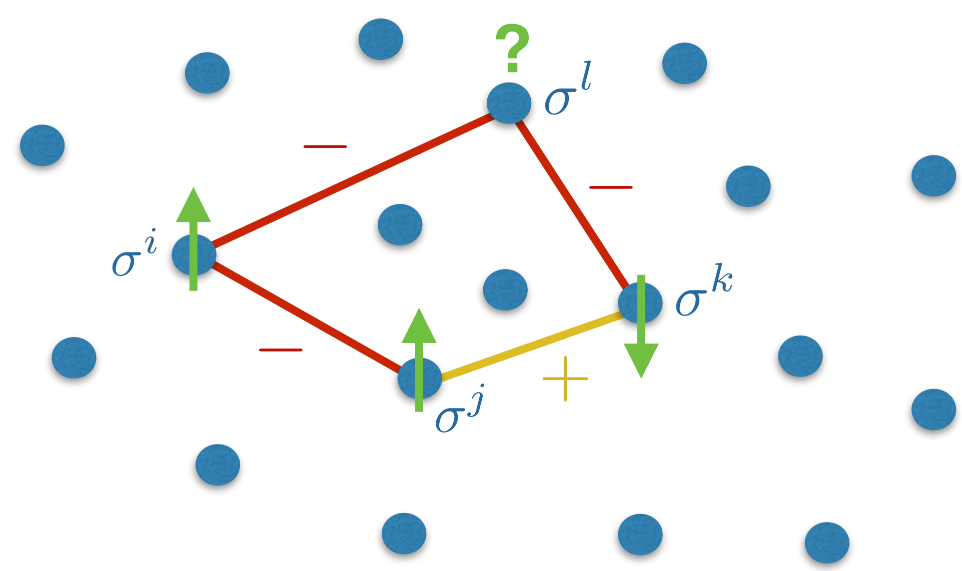
- Many disordered systems exhibit a spin glass phase at low temperatures

- Random couplings lead to frustration, complicated free energy landscape

e.g.: $$H_{\rm SK} = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{ij} \sigma^i \sigma^j$$

$$(\sigma^i = \pm 1\,, \quad \overline{J_{ij}} = 0\,, \quad \overline{J_{ij}^2} = J^2)$$

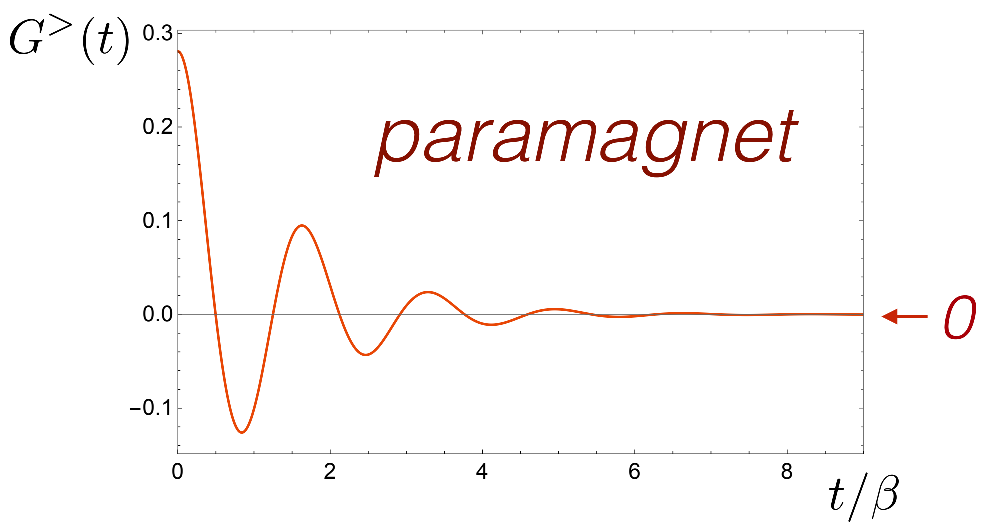


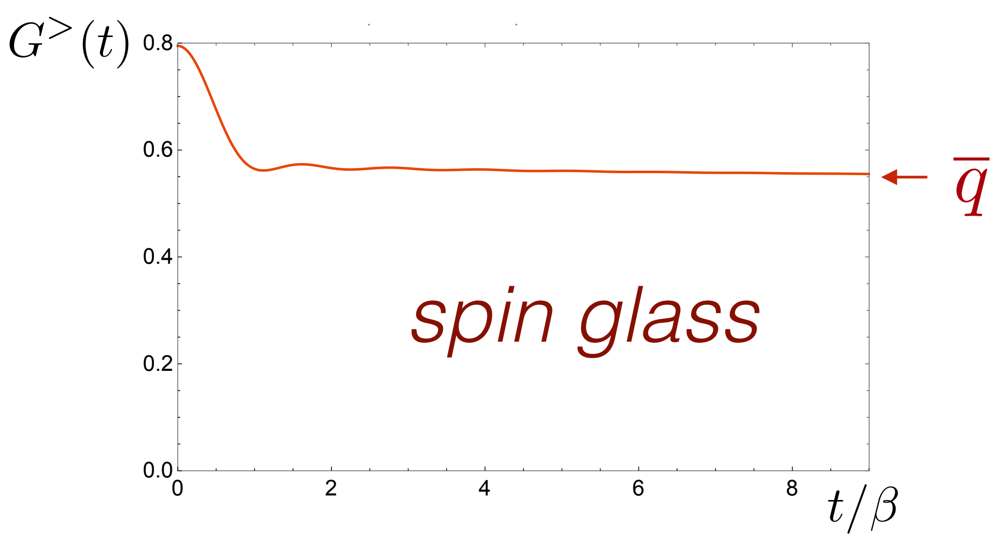




- Many metastable spin glass states separated by high barriers

# Spin glasses

- To detect spin glass phase, consider 'temporal' order parameter




figures: p-spin model [FH/Anous '21]

Averaged two-point function:

$$G^>(t) = \frac{1}{N}\sum_{i=1}^{N} \overline{\langle \sigma^i(t)\sigma^i(0)\rangle}$$

Spin glass order parameter:

$$\overline{q} \equiv \lim_{t\to\infty} G^>(t) = \frac{1}{N}\sum_i \overline{\langle \sigma^i\rangle^2}$$

[Edwards/Anderson '75]

paramagnetic phase: $\overline{q} = 0$

spin glass phase: $\overline{q} > 0$

# Goals

- Characteristic features of SG phase: many metastable states, slow dynamics, inability to reach equilibrium, loss of ergodicity, …

- Understand low temperature properties (with $AdS_2$ gravity in mind):

  - Fate of zero temperature entropy?

  - Fate of reparametrization symmetry?

  - Behavior of Lyapunov exponent?

- Understand within the $nAdS_2/nCFT_1$ paradigm

# Overview

I will mention two (quantum) generalizations of $H_{\text{SK}} = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{ij} \sigma^i \sigma^j$

## SU(M) Heisenberg magnet

$$H = \frac{1}{\sqrt{NM}} \sum_{i<j=1}^{N} J_{ij}\, \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j$$

SU(M) spin operators

- Physical & closely related to (complex) SYK model
- Can see weak spin glass order in fermionic representation

## p-spin spherical model

$$H = \sum_{i_1<\ldots<i_p} J_{i_1\ldots i_p} \sigma^{i_1} \cdots \sigma^{i_p}$$

bosonic 'rotors': $\frac{1}{N} \sum_{i=1}^{N} \sigma^i \sigma^i = M$

- Motivated by higher dimensional constructions
- Two dimensionless parameters: $\beta J,\ MJ$ (thermal & quantum fluctuations)

# Random SU(M) Heisenberg magnet

[Christos/**FH**/Sachdev 2110.00007]

# Random SU(M) Heisenberg magnet

$$H = \frac{1}{\sqrt{NM}} \sum_{i<j=1}^{N} \sum_{\alpha,\beta=1}^{M} J_{ij}\, S^{\alpha}_{\beta}(i) S^{\beta}_{\alpha}(j)$$

[Bray/Moore '80]
[Sachdev/Ye '93]

SU(M) operators on sites *i, j*

- Fermionic spinon representation: $S^{\alpha}_{\beta} = f^{\dagger}_{\beta} f^{\alpha} - \frac{1}{2}\, \delta^{\alpha}_{\beta}$ $\quad (f^{\dagger}_{\alpha} f^{\alpha} = M/2)$
- U(1) gauge invariance: $f^{\alpha}(\tau) \to e^{i\phi(\tau)}\, f^{\alpha}(\tau)$

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Basic idea:
- We always take **large N**
- For $\mathbf{M} \to \infty$: find equations of (complex) SYK
- For **finite** $\mathbf{M}$: (weak) spin glass order

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- Fermionic spinon representation: $S^{\alpha}_{\beta} = f^{\dagger}_{\beta} f^{\alpha} - \frac{1}{2}\,\delta^{\alpha}_{\beta}$ $\quad (f^{\dagger}_{\alpha} f^{\alpha} = M/2)$
- U(1) gauge invariance: $f^{\alpha}(\tau) \to e^{i\phi(\tau)}\, f^{\alpha}(\tau)$
- Want to compute disorder averaged (quenched) free energy:

$$\beta\mathcal{F} = -\overline{\log \mathcal{Z}}$$

$$\mathcal{Z}[J_{ij}] = \int Df^{\alpha}\, D\lambda\, \exp\left\{ -\int d\tau \left[ \sum_i f^{\dagger}_{\alpha}(i)\partial_{\tau} f^{\alpha}(i) + \frac{1}{\sqrt{NM}} \sum_{i,j} J_{ij}\, S^{\alpha}_{\beta}(i) S^{\beta}_{\alpha}(j) \right] - i \int d\tau \sum_i \lambda(i) \left( f^{\dagger}_{\alpha}(i) f^{\alpha}(i) - M/2 \right) \right\}$$

# Replica trick

- Strategy: use replica trick

$$\log \mathcal{Z} = \lim_{n\to 0} \partial_n \mathcal{Z}^n \qquad \Rightarrow \qquad \beta \mathcal{F} = -\lim_{n\to 0} \partial_n \overline{\mathcal{Z}^n}$$

$$\{f^\alpha,\, \lambda\} \longrightarrow \{f^\alpha_a,\, \lambda_a\} \qquad (a = 1, \dots, n)$$

- Introduce bilocal spinon collective field (with replica indices):

$$G_{ab}(\tau, \tau') = -\frac{1}{M} \sum_\alpha f^\alpha_a(\tau) f^\dagger_{b\alpha}(\tau')$$

… and the spinon self-energy $\Sigma_{ab}(\tau, \tau')$

… and a Hubbard-Stratonovich field $Q_{ab}(\tau)$

- **Large N** effective action for $G$, $\Sigma$, $Q$ :

$$\frac{\mathcal{S}[Q]}{N} = \frac{J^2 M}{4} \int d\tau d\tau' \left[ \sum_{a,b} Q_{ab}(\tau - \tau')^2 \right] - \log \mathcal{Z}_f$$

with 'single-site' partition function:

$$\mathcal{Z}_f[Q] = \exp\left( -\frac{k^2 J^2}{2} \int d\tau d\tau' \sum_{a,b} Q_{ab}(\tau, \tau') \right) \int \mathcal{D}G_{ab} \mathcal{D}\Sigma_{ab} \mathcal{D}\lambda_a \;\; e^{-M\, I[Q]}$$

$$I[Q] = -\log\det\left\{ -\delta_{ab}[\partial_\tau + i\lambda_a(\tau)]\delta(\tau - \tau') - \Sigma_{ab}(\tau, \tau') \right\} - ik \int d\tau \sum_a \lambda_a(\tau)$$
$$+ \int d\tau d\tau' \sum_{a,b} \left[ \frac{J^2}{2} Q_{ab}(\tau - \tau')\, G_{ab}(\tau, \tau') G_{ba}(\tau', \tau) - \Sigma_{ab}(\tau, \tau') G_{ba}(\tau', \tau) \right]$$

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- Large N saddle point equation:

$$Q_{ab}(\tau - \tau') = -\frac{k^2}{M} - \frac{1}{\mathcal{Z}_f[Q]} \int \mathcal{D}G_{ab} \mathcal{D}\Sigma_{ab} \mathcal{D}\lambda_a \; G_{ab}(\tau, \tau') G_{ba}(\tau', \tau)\, e^{-M I[Q]}$$

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- Large M saddle point equations:

$$Q_{ab}(\tau) = -G_{ab}(\tau)G_{ba}(-\tau)$$

$$\Sigma_{ab}(\tau) = J^2 Q_{ab}(\tau) G_{ab}(\tau)$$

$$G_{ab}(i\omega) = [i\omega\delta_{ab} - \Sigma_{ab}(i\omega)]^{-1}$$

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- $G_{ab}$ has to be replica diagonal. Therefore, at large M:

$$Q(\tau) = -G(\tau)G(-\tau)$$

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—> e.o.m. of complex SYK

[Sachdev/Ye '93][Sachdev '15][Gu/Kitaev/Sachdev/Tarnopolsky '19]

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[Sachdev/Ye '93][Sachdev '15][Gu/Kitaev/Sachdev/Tarnopolsky '19]

- The solution is a gapless 'fractionalized' spin liquid exhibiting the **SYK physics** that has been explored since [Sachdev/Ye '93] and all its connections to **$AdS_2$ gravity.**

# Infinite vs. finite M

$$Q(\tau) = -G(\tau)G(-\tau)$$
$$\Sigma(\tau) = -J^2 G(\tau)^2 G(-\tau)$$
$$G(i\omega) = [i\omega - \Sigma(i\omega)]^{-1}$$

- Spin-spin spectral function: $Q(\tau) = \int_0^\infty \frac{d\omega}{\pi} \chi''(\omega)\, e^{-\omega\tau}$

$M \to \infty$ :

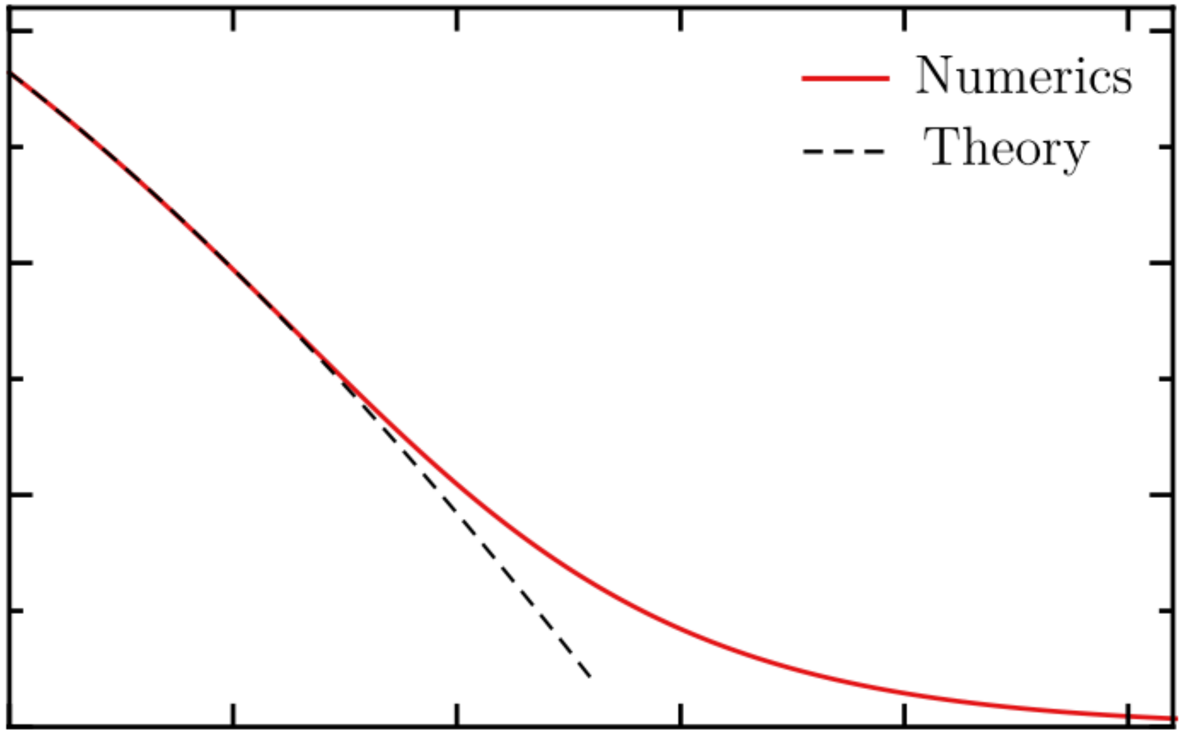
$$\chi''(\omega) \sim \frac{\mathrm{sgn}(\omega)}{J} \left[1 - \frac{c}{J}|\omega| - \ldots\right]$$

'conformal' long-time behavior ($Q(\tau) \sim 1/|\tau|$)

[Sachdev/Ye '93]

'Schwarzian' correction

$\chi''(\omega)$




$\omega$

figure: [Tikhanovskaya/Guo/Sachdev/Tarnopolsky '20]

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[Sachdev/Ye '93]

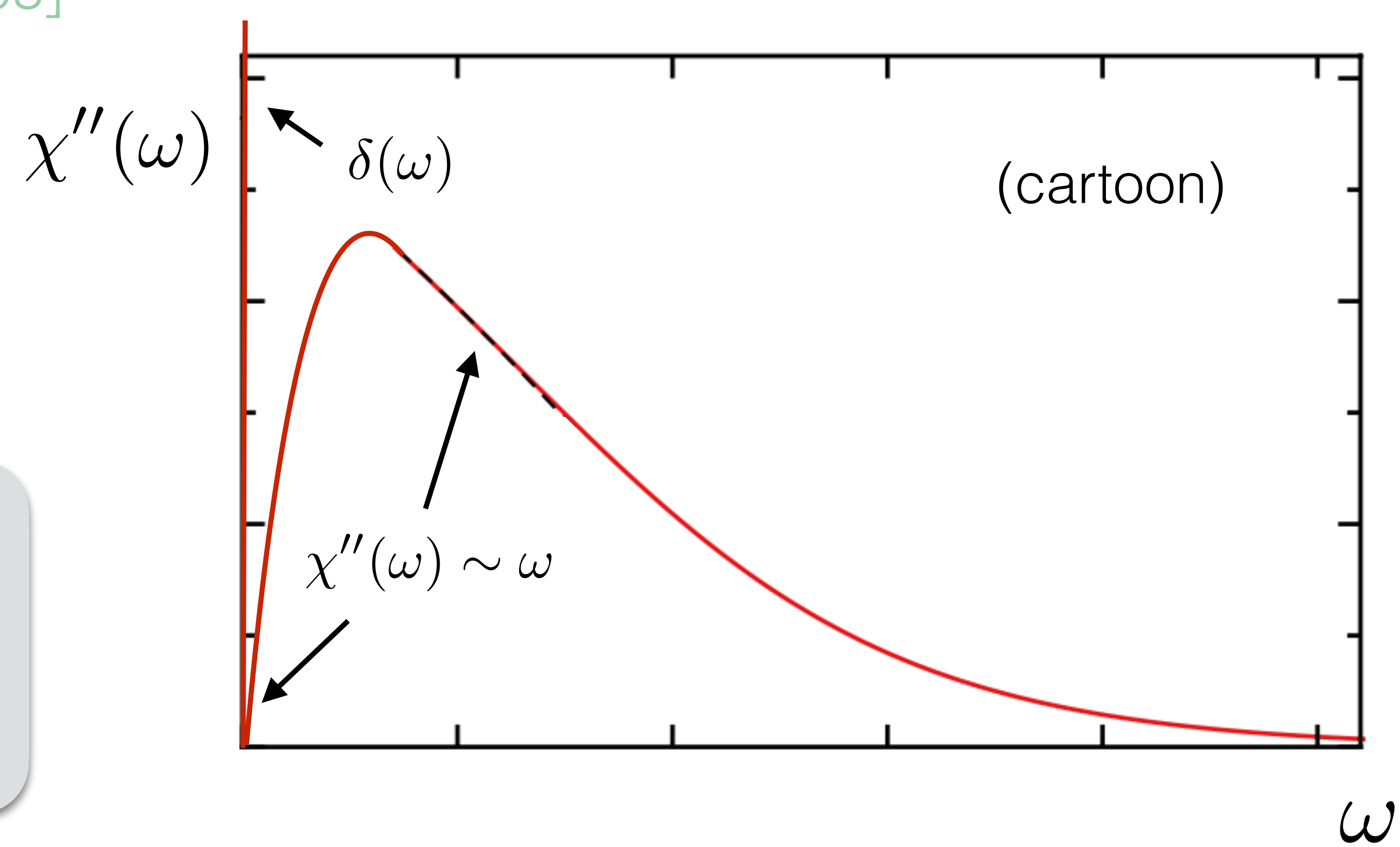
'Schwarzian' correction

- Finite *M* behavior looks different:

[Arrachea/Rozenberg '02]

[Shackleton/Wietek/Georges/Sachdev '21]

> Can we understand this **analytically** in the simple **fermionic** description?



# Finite M corrections

- Finite *M:* $G_{ab}, \Sigma_{ab}, Q_{ab}$ need not be replica diagonal

- Ansatz: $Q_{ab}(\tau) = [Q(\tau) + \overline{q}]\, \delta_{ab} + q_{ab}$ $\qquad (q_{aa} = 0)$

spin glass order parameters

- Compute large-N effective action perturbatively in $\overline{q}, q_{ab}$:

$$\frac{\mathcal{S}[Q(\tau), \overline{q}, q_{ab}]}{nNM} = \frac{(\beta J)^2}{4} \left( \overline{q}^2 + \frac{1}{n} \sum_{a \neq b} q_{ab}^2 \right) \left[ 1 - \frac{J^2}{M} \hat{Q}(\omega = 0)^2 \right] + \ldots$$

onset of spin glass order: $\quad 1 = \frac{J^2}{M} \hat{Q}(0)^2 \quad \Leftrightarrow \quad T_c \sim J\, e^{-\sqrt{M\pi}}$

# Consistency of free energy

- Free energy contains terms that seem divergent as $\beta \to \infty$:

$$\mathcal{F} \equiv -\frac{\ln \mathcal{Z}}{\beta n} = -c_0 - c_1 \overline{q} - c_2 \overline{q}^2 - d_2 \beta \left( \overline{q}^2 + \frac{1}{n} \sum_{a \neq b} q_{ab}^2 \right) - c_3 \overline{q}^3 - e_3 \beta^2 \left( \overline{q}^3 + 3\overline{q} \frac{1}{n} \sum_{a \neq b} q_{ab}^2 + \frac{1}{n} \mathrm{Tr}\, q_{ab}^3 \right)$$
$$- c_4 \overline{q}^4 - d_4 \beta \left( \overline{q}^4 + \frac{1}{n} \sum_{a \neq b} q_{ab}^4 \right) + \dots$$

- But: extremization w.r.t. $\overline{q}, q_{ab}$ makes all dangerous terms vanish!

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- *T>0:* replica symmetry breaking Parisi solution:

$$q_{ab} = \begin{pmatrix} \boxed{A_{m_1}} & & & & \\ & \boxed{A_{m_1}} & & q_0 & \\ & & \ddots & & \\ & q_0 & & \boxed{A_{m_1}} & \\ & & & & \boxed{A_{m_1}} \end{pmatrix}$$

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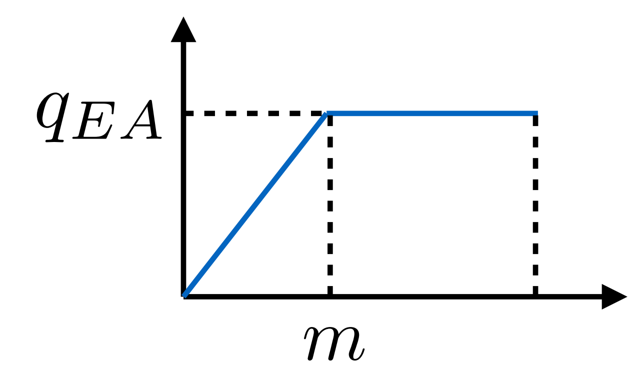
# Parisi solution

- *T>0:* replica symmetry breaking Parisi solution: [Parisi ’79]

$$q_{ab} = \begin{pmatrix} \boxed{A_{m_1}} & & & q_0 & \\ & \boxed{A_{m_1}} & & & \\ & & \ddots & & \\ & q_0 & & \boxed{A_{m_1}} & \\ & & & & \boxed{A_{m_1}} \end{pmatrix}, \quad A_{m_1} = \begin{pmatrix} \boxed{A_{m_2}} & & & q_1 & \\ & \boxed{A_{m_2}} & & & \\ & & \ddots & & \\ & q_1 & & \boxed{A_{m_2}} & \\ & & & & \boxed{A_{m_2}} \end{pmatrix}, \quad A_{m_2} = \dots$$

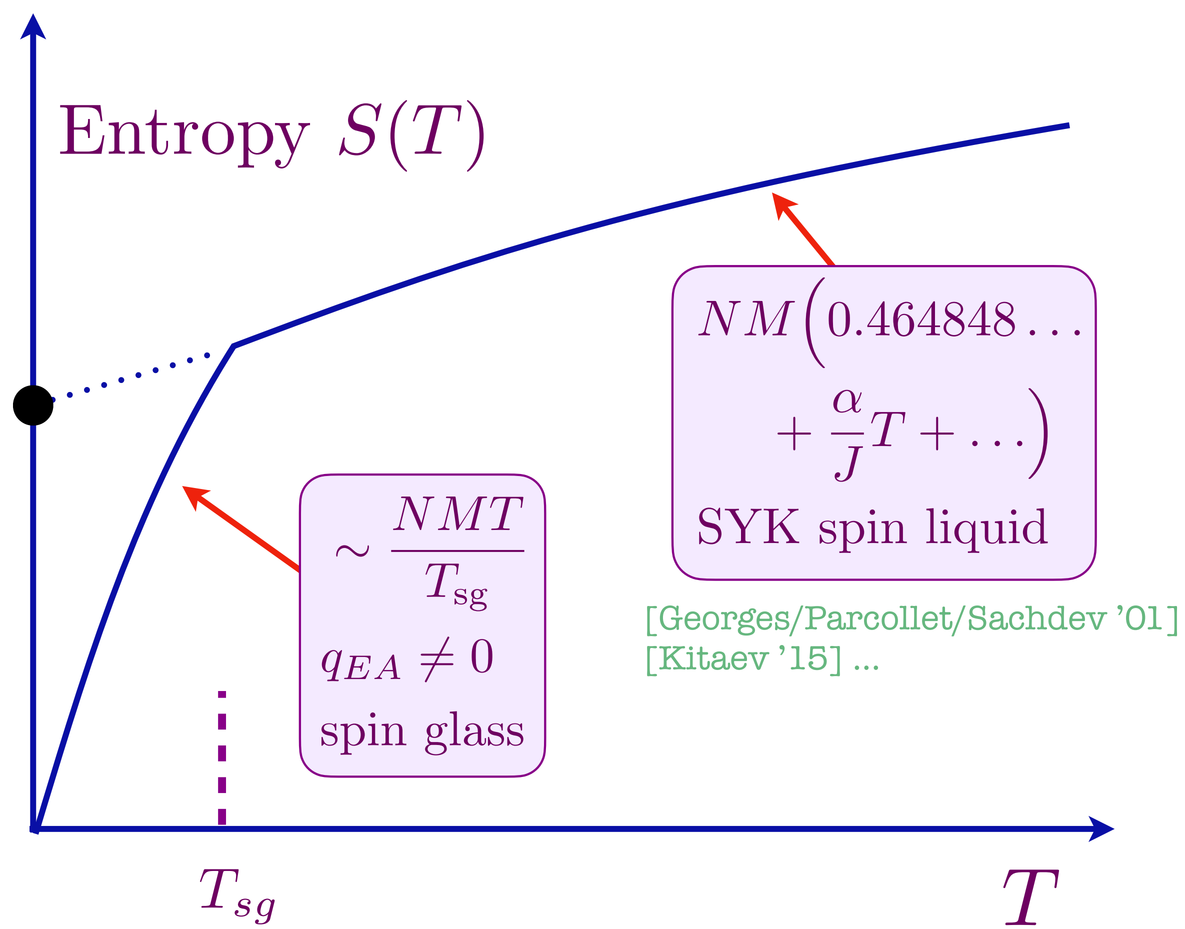
- Limiting case: $q_{ab} \longrightarrow q(x) = \begin{cases} \dfrac{x}{m} q_{EA} & 0 \leq x < m \\ q_{EA} & m \leq x < 1 \end{cases}$




- Useful to think of ‘break point parameter’ *m* as a thermodynamic quantity (similar to $\beta$), characterizing the thermodynamic state

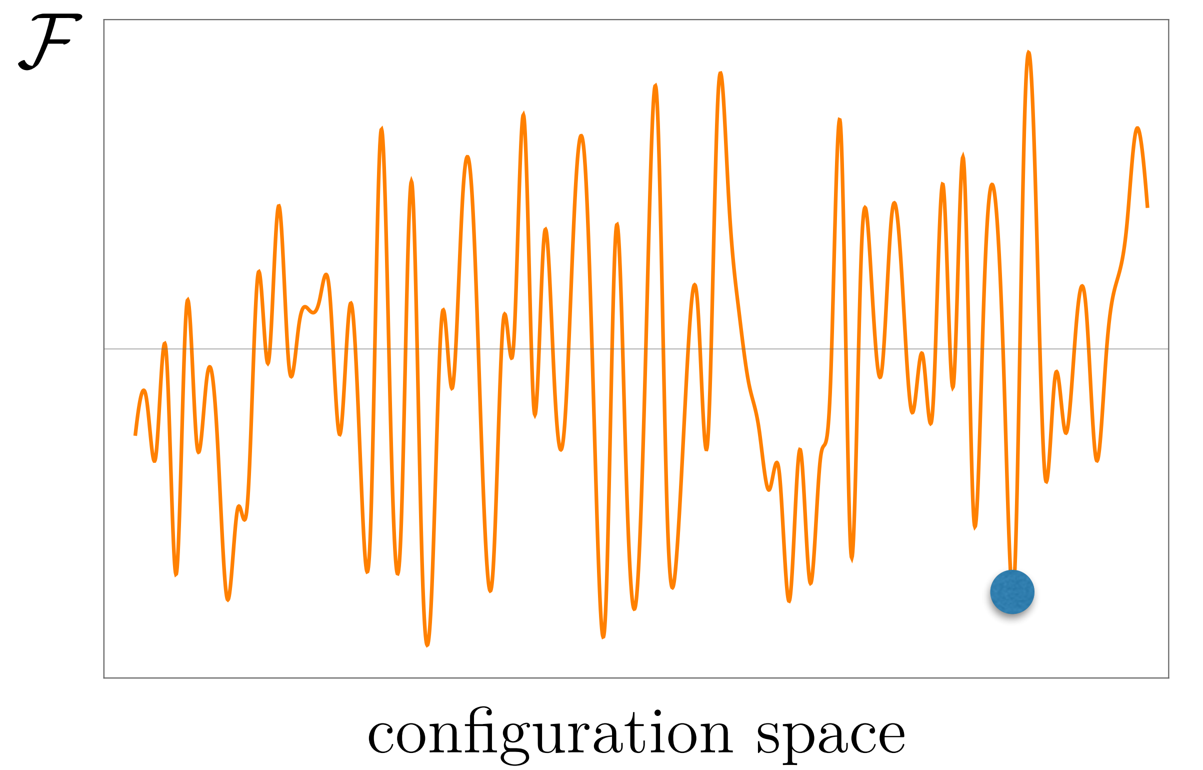
# A puzzle about the entropy

- $\chi''(\omega) \sim \omega \quad \Rightarrow$ no extensive entropy at *T=0* (unlike SYK)




> **Q:** How can the **entropy vanish** if there are exponentially many states?

# A puzzle about the entropy




▸ Q: How can the **entropy vanish** if there are exponentially many states?

Answer:

- System is stuck in a valley.
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- We should also count the number of valleys around $\mathcal{F}$ !

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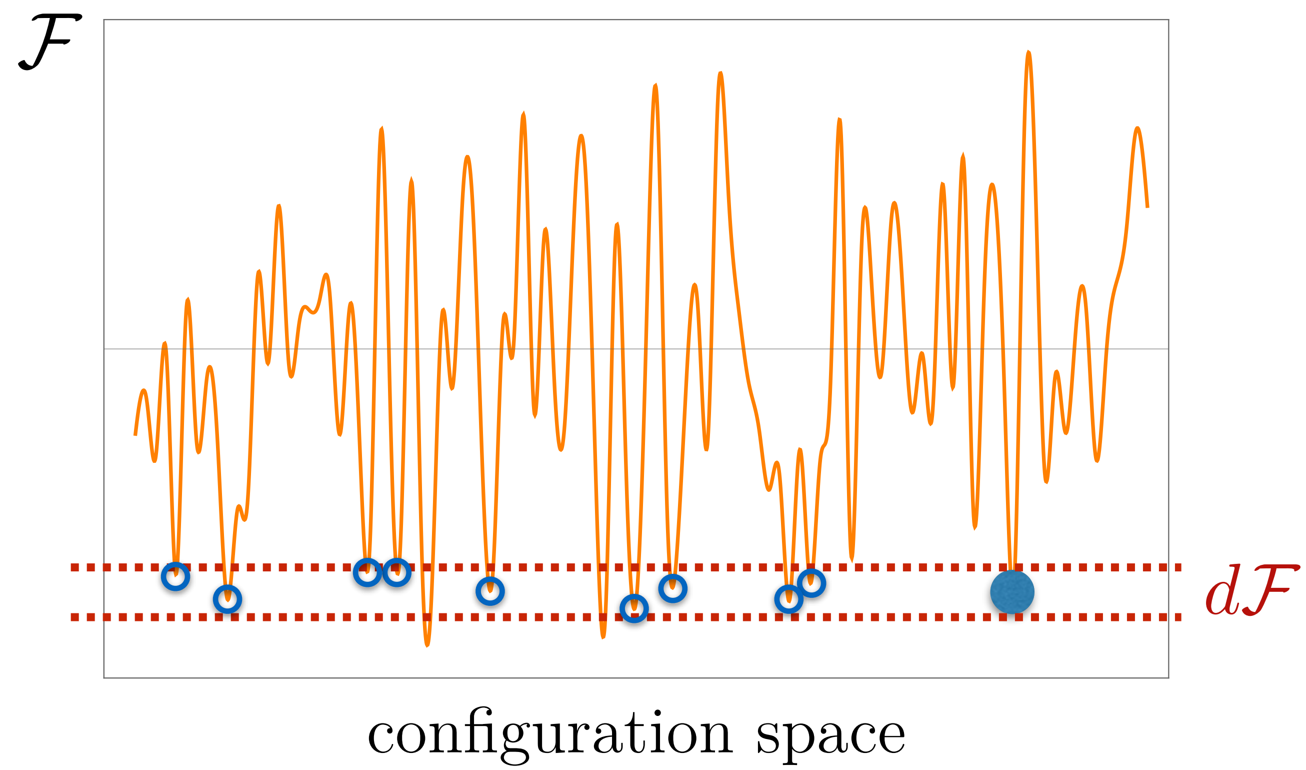


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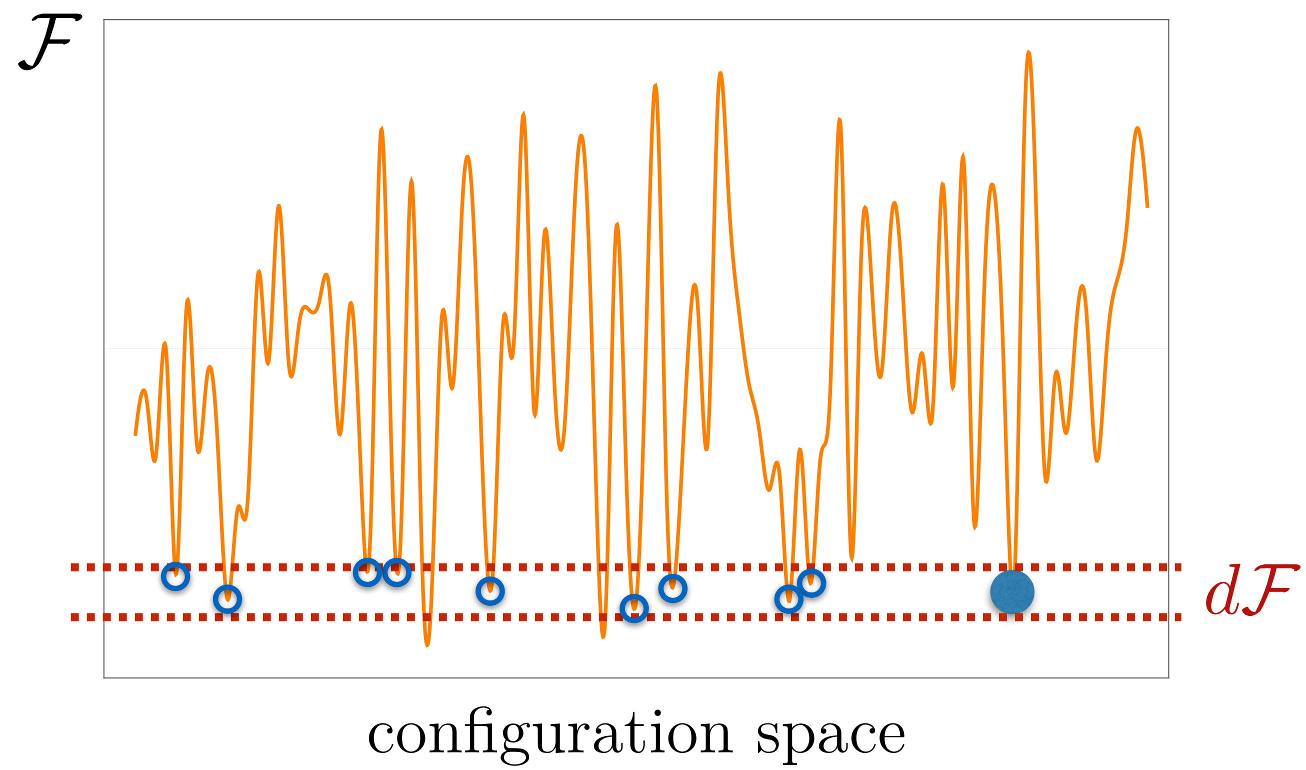
# Complexity

- Define **density of pure states** at free energy $\mathcal{F}$ :

$$\Omega(\mathcal{F}, \beta, N) \equiv e^{N\Sigma(\mathcal{F},\beta)}$$

$\Sigma(\mathcal{F}, \beta)$ : 'complexity'




- Total partition function:

$$Z = e^{-N\beta\mathcal{F}_{\rm tot}} = \int d\mathcal{F}\, e^{-N\beta\mathcal{F}}\, \Omega(\mathcal{F}, \beta, N) = \int d\mathcal{F}\, e^{-N\beta\Phi(\mathcal{F})}$$

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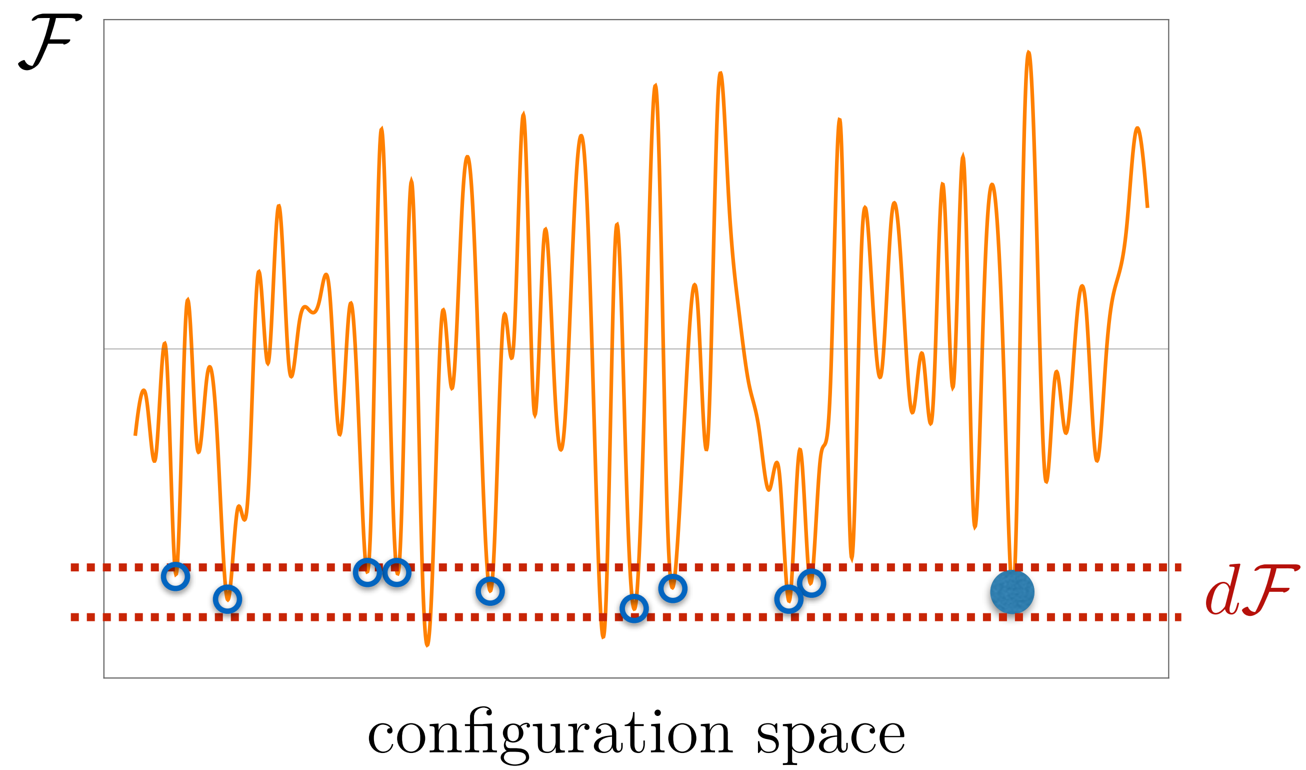


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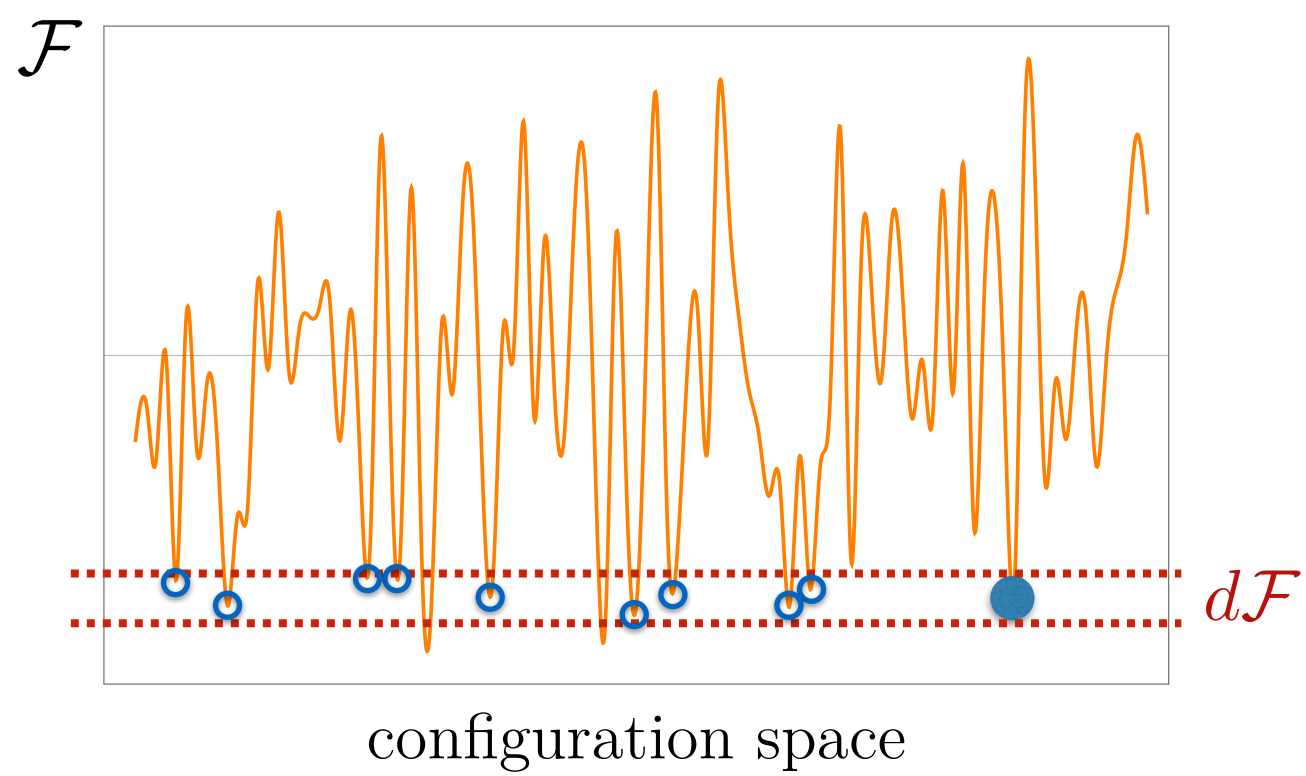


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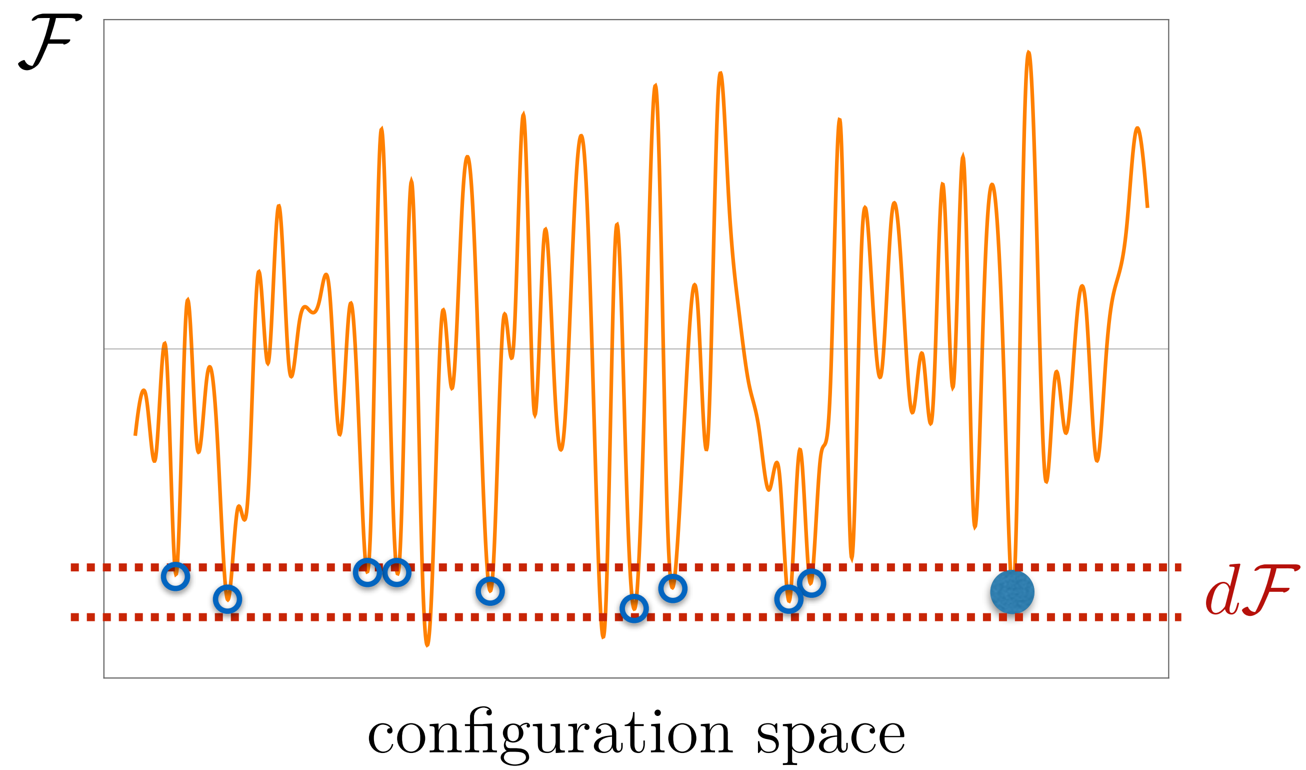


- Using replicas, one can show:

$$\Sigma = \beta m^2 \partial_m \mathcal{F}(m, \beta)$$

c.f. standard thermodynamics: $S = \beta^2 \partial_\beta \mathcal{F}$

[Monasson ’95] [Franz/Parisi ’98] [Mezard/Parisi ’99]

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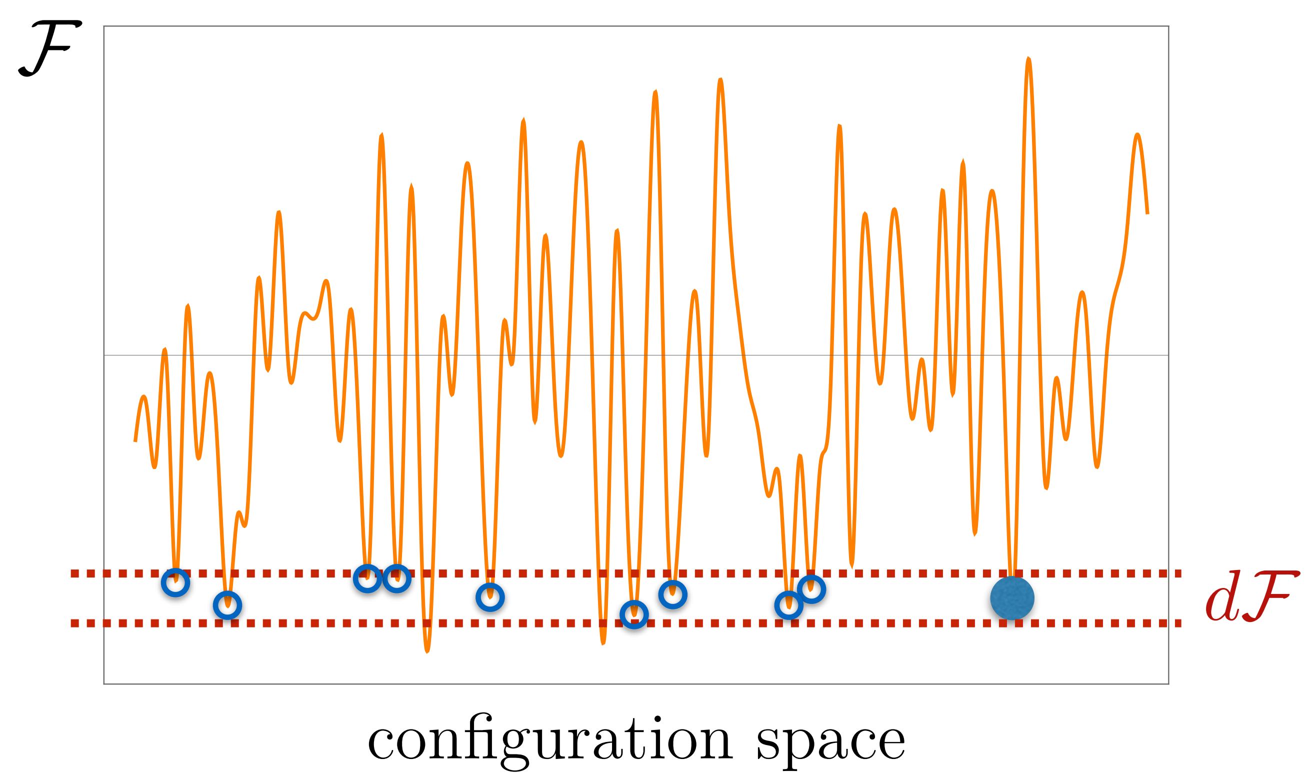


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- In the random Heisenberg magnet, we find:

$$\Sigma = \frac{12 d_4}{e_3^2} q_{EA}^2 \left(d_2 + 6 d_4 q_{EA}^2 + \ldots\right)^2 + \mathcal{O}(\beta^{-1})$$

In the low temperature spin glass ( $q_{EA} \neq 0$ ) the extensive entropy of SYK gets replaced by an extensive complexity.

# Holographic speculations

# Holographic glasses

- The following is a suggestion (haven't worked out details) inspired by earlier work: [Anninos/Anous/Barandes/Denef/Gaasbeek '11] [Anninos/Anous/Denef/Konstantinidis/Shaghoulian '12] [Anninos/Anous/Denef/Peeters '15] [Anninos/Anous/Denef '16] [Anous/FH '21]

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> - How to accommodate for an extensive landscape of spin glass states in holography?

- Hint: in 4d N=2 supergravity, there exists a landscape of fragmented **multi-centered black holes**
  [Denef '00] [Cardoso/Wit/Kappeli/Mohaupt '00] [Bates/Denef '11]

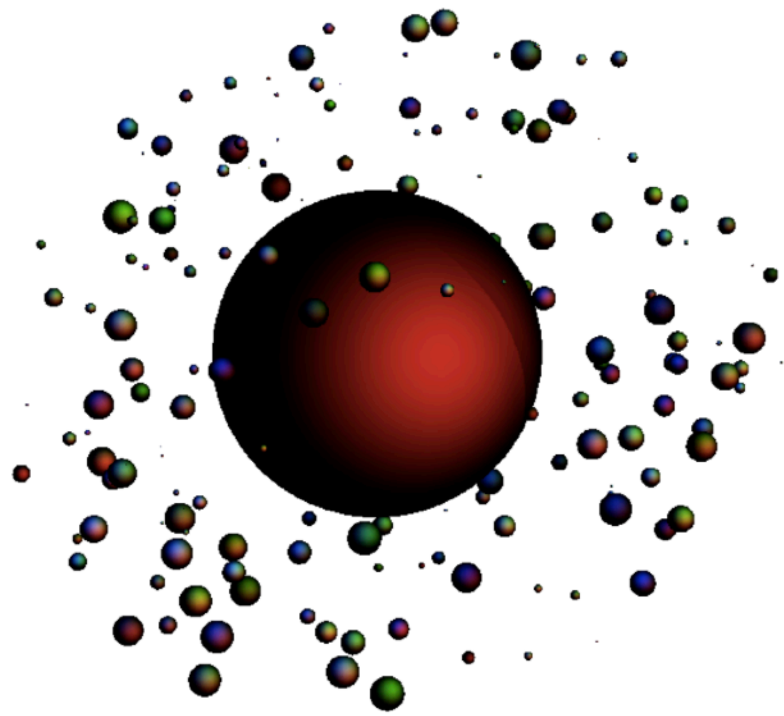


figure: [Anninos/Anous/Barandes/Denef/Gaasbeek '11]

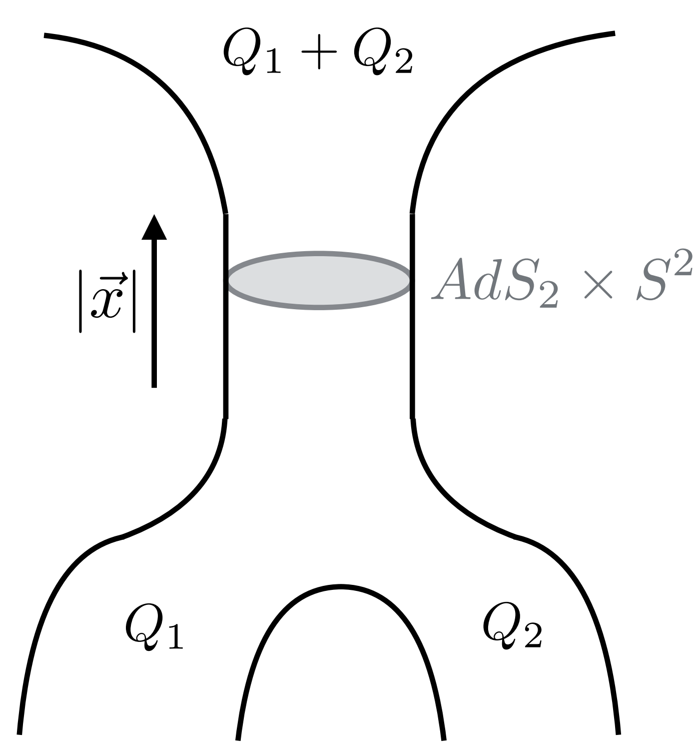
# AdS$_2$ fragmentation

- Near horizon of 4d extremal two-RN black holes with fixed charge:

$$ds^2 = -V^{-2}dt^2 + V^2 d\vec{x}^2$$

$$\star F = dt \wedge dV^{-1}$$

$$V = \frac{Q_1}{|\vec{x}-\vec{x}_1|} + \frac{Q_2}{|\vec{x}-\vec{x}_2|}$$




[Majumdar/Papapetrou '47]

[Brill '92]

[Maldacena/Michelson/Strominger '98]

- Large $|\vec{x}|$ : same as geometry with a single throat with charge $Q_1 + Q_2$
- For $\vec{x} \to \vec{x}_{1,2}$ : fragmentation into two (or more) AdS$_2$ regions

# AdS₂ fragmentation

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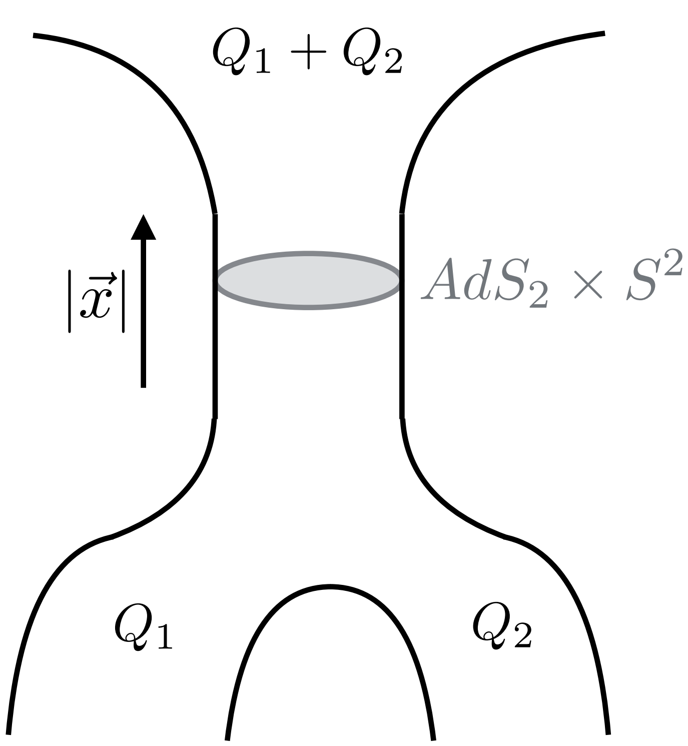



[Majumdar/Papapetrou '47]

[Brill '92]

[Maldacena/Michelson/Strominger '98]

- Moduli space of geometries, which locally minimize free energy

  ⟶ related to complexity of spin glass landscape?

- Non-zero average dipole moment

  ⟶ related to order parameter $q_{EA}$?

[Anous/FH '21]

# The p-spin spherical model

[Anous/**FH** 2106.03838]

# Brane constructions

- Certain string theory compactifications with D-branes lead to quiver quantum mechanics with a sector similar to a disordered system:
  - Chiral and vector multiplets
  - SUSY constrains structure of Lagrangian —> bosonic potential:

$$V = \sum_{i,a} \left| \frac{\partial W(\phi)}{\partial \phi_i^a} \right|^2 + \frac{1}{g_{\rm YM}^2} \sum_a \left( \theta_a - \sum_i |\phi_i^a|^2 \right)^2$$

[Douglas/Moore '96]
[Denef/Moore '07]

superpotential
$W(\phi) = \Omega_{ijk}\phi_i^1\phi_j^2\phi_k^3 + \dots$

FY parameters

▷ Consider a toy model, which resembles this structure!

# The p-spin spherical model

$P(J_{i_1 \ldots i_p}) \propto \exp\left[-\frac{N^{p-1}}{p!}\frac{J^2_{i_1\ldots i_p}}{J^2}\right]$

$$Z[J_{i_1 \ldots i_p}] = \int D\sigma_i Dz \ \exp\left\{ -\int_0^\beta d\tau \left[\frac{M}{2}\dot{\sigma}_i(\tau)\dot{\sigma}_i(\tau) + \sum_{i_1<\ldots<i_p} J_{i_1\ldots i_p}\sigma_{i_1}(\tau)\ldots\sigma_{i_p}(\tau)\right] + i\int_0^\beta d\tau\, z(\tau)\left(\sum_{i=1}^N \sigma_i(\tau)\sigma_i(\tau) - N\right)\right\}$$

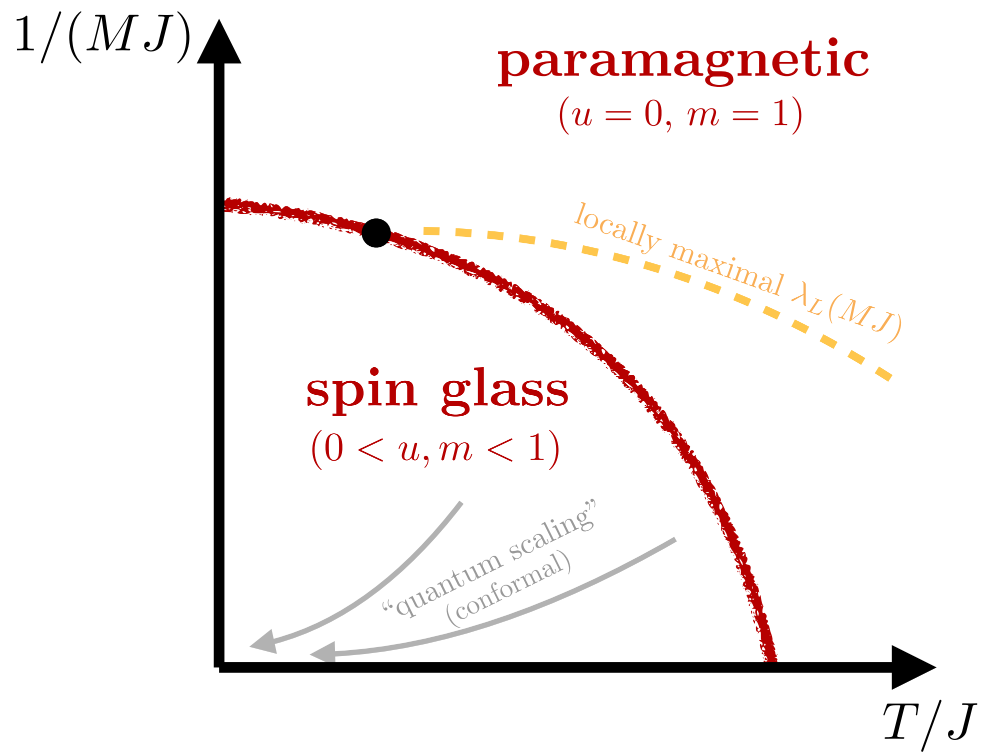
"spherical constraint"

- Dimensionless parameters: $\beta J, \ MJ$
- Nonlinear sigma-model with spherical target space
- Spherical constraint is crucial for stability of such a bosonic model

[Crisanti/Sommers '92] [Cugliandolo/Grempel/da Silva Santos '01]

# p-spin model: summary of features

- Phase diagram:




- Spin glass order requires both: small thermal and quantum fluctuations

- Spin glass order is strong ( $u \to 1$ as $T \to 0$ )

- Strong coupling (large $\beta J$ and $MJ$ ):
  - Complexity is again finite and extensive: $\Sigma = \frac{1}{2}\log(p-1) - \frac{p-2}{p}$
  - Gapless spectrum $\sim \omega$, power-law scaling, reparametrization invariance
  - Non-zero but suppressed Lyapunov exponent: $\lambda_L \sim \frac{2\pi}{\beta}(p-2)\left[\frac{5}{24\beta\mathcal{J}} + \dots\right]$

[Anous/FH '21] [Bera/Lokesh/Banerjee '21]

# Summary

# Summary

## SU(M) Heisenberg magnet

$$H = \frac{1}{\sqrt{NM}} \sum_{i<j=1}^{N} J_{ij}\, \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j$$

SU(M) spin operators

- Physical & closely related to (complex) SYK model
- Weak spin glass order in fermionic representation
- Analytical treatment of transition from 'deconfined' SYK spin liquid to 'confined' spin glass

## p-spin spherical model

$$H = \sum_{i_1<\ldots<i_p} J_{i_1\cdots i_p} \sigma^{i_1} \cdots \sigma^{i_p}$$

bosonic 'rotors': $\frac{1}{N}\sum_{i=1}^{N} \sigma^i \sigma^i = M$

- Two-dimensional parameter space: $\beta J,\ MJ$
- Analytically tractable in strongly coupled spin glass
- Intricate dependence of $\lambda_L$ on couplings

AdS2 fragmentation: interpretation of complexity, order parameters, … ?