Adiabatic Hydrodynamics and the Eightfold Way to Dissipation

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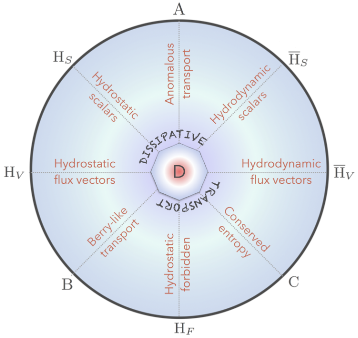

Perimeter Institute — 12 November 2015

FH, R. Loganayagam, M. Rangamani

[1412.1090], [1502.00636], **(see also [1312.0610], [1510.02494], [1511.xxxxx], ...)**

Outline

- **The hydrodynamic gradient expansion**
- Classification of transport
- Example: conformal fluid
- Schwinger-Keldysh and emergent gauge symmetry
- Outlook and conclusion

The hydrodynamic gradient expansion

- Hydrodynamics: generic near-equilibrium eff. field theory for long wavelength fluctuations

microscopic theory

$$\downarrow L \gg \ell_{\mathsf{mfp}}$$

macroscopic fluid variables:	$u^\mu(x),\ T(x),\ \mu(x)$	$(u^2=-1)$
background sources:	$g_{\mu\nu}(x),\ A_\mu(x)$	

$$\downarrow$$

Constitutive relations:	Dynamics:
$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \ldots$	$\nabla_\nu T^{\mu\nu} \simeq F^{\mu\nu} J_\nu$
$J^\alpha = J^\alpha_{(0)} + J^\alpha_{(1)} + \ldots$	$\nabla_\alpha J^\alpha \simeq 0$

- E.g.: $T^{\mu\nu}_{(0)} = \varepsilon(T,\mu)\, u^\mu u^\nu + p(T,\mu)\, (g^{\mu\nu} + u^\mu u^\nu)\,, \qquad J^\alpha_{(0)} = q(T,\mu)\, u^\alpha$

The hydrodynamic gradient expansion

- Hydrodynamics: generic near-equilibrium eff. field theory for long wavelength fluctuations

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$\downarrow$

Constitutive relations:	Dynamics:
$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \ldots$	$\nabla_\nu T^{\mu\nu} \simeq F^{\mu\nu} J_\nu + \boxed{\mathrm{T}^{\perp\nu}_{H}}$
$J^\alpha = J^\alpha_{(0)} + J^\alpha_{(1)} + \ldots$	$\nabla_\alpha J^\alpha \simeq \boxed{\mathrm{J}^{\perp}_{H}}$ (cov. anomalies)

- E.g.: $T^{\mu\nu}_{(0)} = \varepsilon(T,\mu)\, u^\mu u^\nu + p(T,\mu)\,(g^{\mu\nu} + u^\mu u^\nu)\,, \qquad J^\alpha_{(0)} = q(T,\mu)\, u^\alpha$

The hydrodynamic gradient expansion

- **"Solving hydrodynamics"** in this talk does not mean...
  ... to calculate transport coefficients $(\varepsilon, p, \ldots)$ for any particular microscopic system
  ... to solve the fluid equations for $\{u^\mu(x), T(x), \mu(x)\}$
- It rather means: **to provide all symmetry-allowed constitutive relations order by order in ∇_μ which are consistent with the**

Second law constraint

$\exists\ J_S^\mu = s(T,\mu)\, u^\mu + J_{S,(1)}^\mu + \ldots$ with $\nabla_\mu J_S^\mu \gtrsim 0$ (on-shell)

- Gives quite non-trivial constraints on physically allowed constitutive relations, e.g.:
  - Neutral ideal fluid: $\varepsilon + p = s\,T$
  - Neutral 1st order: viscosities $\eta, \zeta \geq 0$
  - Neutral 2nd order: 5 relations among 15 a-priori independent transport coefficients
  - Anomaly induced transport completely fixed

Bhattacharyya '12

Son-Surowka '09

Jensen-Loganayagam-Yarom '13

⋮

Plan

This talk: the **structure** of hydrodynamics

- Understand constitutive relations allowed by Second Law:
  - **Classify** hydrodynamic transport in a physically useful way
  - Construct **most general solution** at all orders
- Suggest a **unifying framework** for adiabatic transport:
  - **New gauge symmetry** that explains the 2nd law constraint

Talk on Nov 20: the **field theory** behind hydrodynamics

- Use hydrodynamics as a tractable starting point to learn basic lessons about some important problems across physics:
  - How to understand hydrodynamics as a **Wilsonian field theorist?**
  - Ingredients: Schwinger-Keldysh, SUSY, topological σ-models, ...
  - **Black holes via AdS/CFT:** dissipation, entropy, unitarity, ...

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- **Classification of transport**
- Example: conformal fluid
- Schwinger-Keldysh and emergent gauge symmetry
- Outlook and conclusion

Disclaimer

From now on I will only discuss **neutral fluids**.
Adding an arbitrary number of abelian or non-abelian flavours is just a technical task without new conceptual ideas, see [1502.00636].

Off-shell entropy production and adiabaticity

- Inequality constraint $\nabla_\mu J_S^\mu \gtrsim 0$ is much more conveniently incorporated if we don't have to simplify it using equations of motion.
- Use Lagrange multiplier $\boldsymbol{\beta}^\mu$ and consider **off-shell statement**:

$$\nabla_\mu J_S^\mu + \boldsymbol{\beta}_\mu \left(\nabla_\nu T^{\mu\nu} - \mathrm{T}_H^{\mu\perp} \right) \equiv \Delta \geq 0$$

Loganayagam '11

- Natural Lagrange multiplier:
  - $\boldsymbol{\beta}^\mu = \frac{1}{T}\, u^\mu$ (local thermal vector)

Task: solve for $\{J_S^\mu, T^{\mu\nu}\}$ as functionals of $\{\boldsymbol{\beta}^\mu, g_{\mu\nu}\}$

▷ Ideally: find effective action which defines all solutions off-shell

- Marginal case $\Delta = 0$: **'adiabaticity equation'**
  - Particularly rich structure! $\Rightarrow$ focus on this first

Aside: adiabaticity equation for free energy current

$$\nabla_\mu J_S^\mu + \boldsymbol{\beta}_\mu \left(\nabla_\nu T^{\mu\nu} - \mathrm{T}_H^{\mu\perp} \right) = 0$$

- Can trade entropy current J_S^μ for **free energy current** $\mathcal{G}^\mu$:

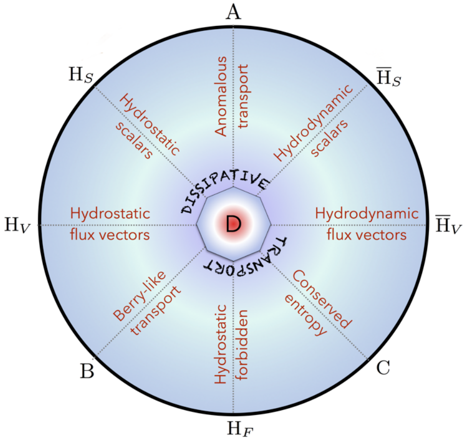
$$-\frac{\mathcal{G}^\mu}{T} \equiv J_S^\mu - (J_S^\mu)_{canonical} \qquad \text{with} \qquad (J_S^\mu)_{canonical} = -\boldsymbol{\beta}_\nu T^{\mu\nu}$$

- Grand-canonical version of adiabaticity equation:

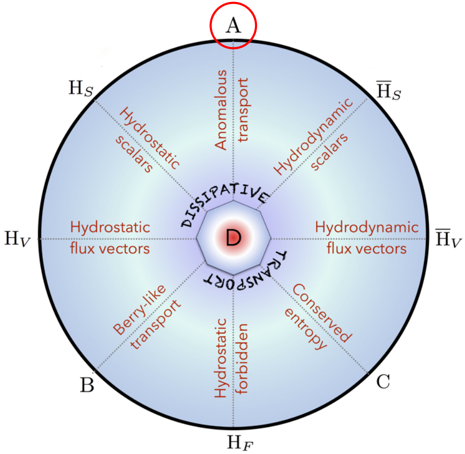
$$-\left[\nabla_\mu \left(\frac{\mathcal{G}^\mu}{T} \right) - \frac{\mathcal{G}_H^\perp}{T} \right] = \frac{1}{2}\, T^{\mu\nu}\, \pounds_{\boldsymbol{\beta}}\, g_{\mu\nu}$$

  - Solve for $\{\mathcal{G}^\mu, T^{\mu\nu}\}$ as functionals of $\{\boldsymbol{\beta}^\mu, g_{\mu\nu}\}$

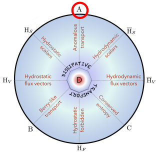
Classification of hydrodynamic transport



Classification of hydrodynamic transport



Anomaly induced transport (Class A)



$$-\left[\nabla_\sigma\left(\frac{\mathcal{G}^\sigma}{T}\right)-\frac{\mathcal{G}^\perp_H}{T}\right]=\frac{1}{2}\,T^{\mu\nu}\,\pounds_{\boldsymbol{\beta}}\,g_{\mu\nu}+\Delta$$

- First of all: let's get rid of anomalies $\mathcal{G}^\perp_H = -u_\nu \mathrm{T}^{\nu\perp}_H$
  - Can always split off from a solution $\{\mathcal{G}^\sigma, T^{\mu\nu}\}$ a **particular solution** $\{(\mathcal{G}^\sigma)_A, (T^{\mu\nu})_A\}$ that takes care of anomalies with $(\Delta)_A = 0$:

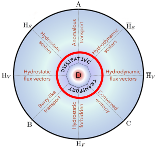
$$-\left[\nabla_\sigma\left(\frac{(\mathcal{G}^\sigma)_A}{T}\right)-\frac{\mathcal{G}^\perp_H}{T}\right]=\frac{1}{2}\,(T^{\mu\nu})_A\,\pounds_{\boldsymbol{\beta}}\,g_{\mu\nu}$$

Loganayagam '11

Jensen-Loganayagam-Yarom '13

  - Anomalous transport coefficients fixed in terms of anomaly polynomial $\Rightarrow$ finite class

Dissipative transport (Class D)

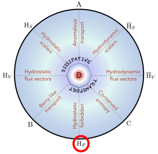


$$\Delta \equiv -\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T} \right) - \frac{1}{2} T^{\mu\nu} \, \pounds_{\boldsymbol{\beta}} \, g_{\mu\nu} \geq 0$$

- Now consider transport which does generically produce entropy ($\Delta > 0$)
- Such terms appear in three varieties:
  1. Sign-definite terms (inequalities from 2nd law)
     → **These only show up at leading order!** *Bhattacharyya '11 '13 '14*
  2. Sign-indefinite terms which are dominated by sign-definite terms (no constraints from 2nd law)
  3. Sign-indefinite terms which are dominant in derivative expansion (forbidden by 2nd law)
- **Example:** $T^{\mu\nu}_{(1)} = -\zeta \, \Theta \, (g^{\mu\nu} + u^\mu u^\nu) \qquad (\Theta \equiv \nabla_\mu u^\mu)$
  gives $\Delta = \zeta \, \frac{1}{T} \Theta^2 \quad \Rightarrow \quad \zeta \geq 0$ (type 1)

  $\Rightarrow$ At higher orders:
  any $T^{\mu\nu}_{(2)} = \gamma \, [\partial\partial]^{\mu\nu}$ s.t. $\Delta = \gamma \, \Theta^3$
  will be subdominant, hence unconstrained (type 2)

Hydrostatically forbidden terms (Class H_F)



$$\Delta \equiv -\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T}\right) - \frac{1}{2} T^{\mu\nu} \, \pounds_{\boldsymbol{\beta}} \, g_{\mu\nu} \geq 0$$

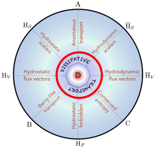
- Type ③: sign-indefinite terms at dominant order in ∂
  - Need to be zero for consistency with 2nd law!
  - **Example:** Ideal fluid

$$T^{\mu\nu}_{(0)} = \varepsilon \, u^\mu u^\nu + p \, (g^{\mu\nu} + u^\mu u^\nu) \,, \qquad J^\mu_{S,(0)} = s \, u^\mu$$

$$\Rightarrow \quad \Delta \simeq \underbrace{(Ts - \varepsilon - p)}_{=0 \text{ (!)}} \, \frac{\Theta}{T} + \underbrace{\left(T \frac{ds}{dT} - \frac{d\varepsilon}{dT}\right)}_{=0 \text{ (!)}} \, \frac{(u\nabla)T}{T}$$

  - A-priori: 3 parameters
  - But second law enforces: 2 relations
- This is **Class** H_F**:** combinations forbidden by 2nd law

Dissipative transport (Class D)



$$\Delta \equiv -\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T}\right) - \frac{1}{2} T^{\mu\nu}\, \pounds_{\boldsymbol{\beta}}\, g_{\mu\nu} \geq 0$$

- There's a simple way to construct Class D explicitly:

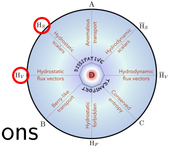
$$(T^{\mu\nu})_{\mathsf{D}} = -\left(\mathcal{N}^{(\mu\nu)(\alpha\beta)} + \mathcal{N}^{(\alpha\beta)(\mu\nu)}\right) \pounds_{\boldsymbol{\beta}}\, g_{\alpha\beta}$$
$$(\mathcal{G}^\sigma)_{\mathsf{D}} = 0$$

  - Every term in Δ involves a total square of the form

$$\Delta = \mathcal{N}^{((\mu\nu)(\alpha\beta))} (\pounds_{\boldsymbol{\beta}}\, g_{\mu\nu})(\pounds_{\boldsymbol{\beta}}\, g_{\alpha\beta})$$

  - Hence $\Delta \geq 0$ automatic as long as the leading order is sign-definite
- Easy task at any order in ∇: find all tensor structures $\mathcal{N}^{(\alpha\beta)(\mu\nu)}[\boldsymbol{\beta}^\mu, g_{\mu\nu}]$

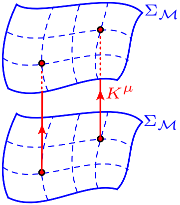
Hydrostatics (Class H)



- Hydrostatic transport: time-independent equilibrium configurations
  - $\exists$ timelike Killing vector $K^\mu = \boldsymbol{\beta}^\mu|_{equil.}$:

    $$\pounds_K \, g_{\mu\nu} = 0$$

  - Spacetime manifold $\mathcal{M}$:
    Euclidean structure $\Sigma_{\mathcal{M}} \times S^1$

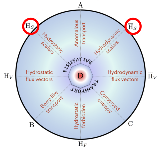


- Transport captured by **Euclidean path integral/partition function:**

$$W_{\text{Hydrostatic}} = -[\text{total free energy}] = -\left[\int_{\Sigma_{\mathcal{M}}} \left(\frac{\mathcal{G}^\sigma}{T}\right) d^{d-1}S_\sigma\right]_{\text{Hydrostatic}}$$

  - Decompose: $\mathcal{G}^\sigma = \mathcal{S}\,\boldsymbol{\beta}^\sigma + \mathcal{V}^\sigma$
  - This splits Class H into two subclasses: $\mathrm{H} = \mathrm{H}_S \cup \mathrm{H}_V$
  - Variation w.r.t. $g_{\mu\nu}$ gives all hydrostatic $T^{\mu\nu}$

Lagrangian solutions (Class L)



- Consider Landau-Ginzburg action:
  - Fields: fluid vector & background geometry $= \{\boldsymbol{\beta}^\mu, \, g_{\mu\nu}\}$
  - Symmetries: diffeomorphism invariance

$$S_{\text{eff}} = \int \sqrt{-g}\; \mathcal{L}[\boldsymbol{\beta}^\mu, g_{\mu\nu}]$$

  - Basic variation defines hydrodynamic currents:

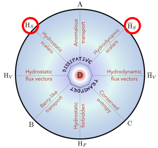
$$\delta S_{\text{eff}} = \int \sqrt{-g}\, \left[\tfrac{1}{2}\, T^{\mu\nu}\, \delta g_{\mu\nu} + T\, \mathfrak{h}_\sigma\, \delta\boldsymbol{\beta}^\sigma + \underbrace{\nabla_\mu(\cdots)^\mu}_{\text{surface term}}\right]$$

  - Further, define entropy current:

$$J_S^\mu = s\, u^\mu \qquad \text{with} \qquad s \equiv \left[\frac{1}{\sqrt{-g}}\frac{\delta S_{\text{eff}}}{\delta T}\right]_{\{u^\mu, g_{\mu\nu}\}\text{ fixed}} = -\mathfrak{h}_\sigma \boldsymbol{\beta}^\sigma$$

  - Can show: $\{T^{\mu\nu}, J_S^\mu\}$ solve **adiabaticity equation** (= off-shell version of 2nd law constraint)

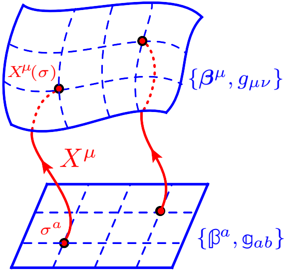
Lagrangian solutions (Class L)



- To get correct dynamics, formulate problem as a σ-**model:**

$$X^\mu : \quad \begin{array}{c}\textbf{worldvolume}\\ \text{reference con-}\\ \text{figuration}\end{array} \longrightarrow \begin{array}{c}\textbf{space filling}\\ \textbf{brane}\\ \text{(physical}\\ \text{fluid)}\end{array}$$

$$\mathfrak{g}_{ab} = \frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b} g_{\mu\nu}[X(\sigma)]\,, \qquad \mathbb{\beta}^a = \frac{\partial \sigma^a}{\partial X^\mu}\boldsymbol{\beta}^\mu[X(\sigma)]$$



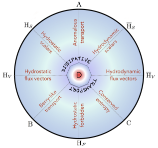
  - Vary pullback fields X^μ, while holding the reference configuration $\mathbb{\beta}^a$ fixed

$$\left.\begin{array}{r}\dfrac{\delta S_{\text{eff}}}{\delta X^\mu} = 0\\ + \text{ diffeo Bianchi id.}\end{array}\right\} \quad \Rightarrow \quad \nabla_\mu T^{\mu\nu} \simeq 0$$

Lesson: fluids are naturally σ-models with dynamical d.o.f. = pullback maps

(c.f. formulation of non-dissipative fluids in terms of **Goldstone modes** *Dubovsky-Hui-Nicolis-Son '11*)

What do we have so far?

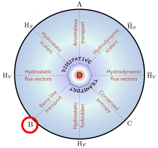


- **Class A:** anomalies can be dealt with once and forever
- **Class D:** can genuinely produce entropy $(\Delta \geq 0)$
- **Class** H_F**:** Constitutive relations forbidden by 2nd law

$$\mathcal{G}^\sigma = \mathcal{S}\boldsymbol{\beta}^\sigma + \mathcal{V}^\sigma$$

- **Class H** $= \mathrm{H}_S \cup \mathrm{H}_V$**:** Hydrostatic response to free energy density and flux
- **Class L:** Wilsonian action giving currents consistent with 2nd law
  - But Lagrangians being scalars, we only get: Class L $= \mathrm{H}_S \cup \overline{\mathrm{H}}_S$
- Some more situations that we're missing so far:
  - Free energy current $\mathcal{G}^\sigma$ could be **zero** or **topological**
  - **Non-hydrostatic** free energy flux vectors $(\overline{\mathrm{H}}_V)$

Berry-curvature type solutions (Class B)



$$\Delta \equiv -\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T}\right) - \frac{1}{2} T^{\mu\nu} \, \pounds_{\boldsymbol{\beta}} \, g_{\mu\nu} = 0$$

- Consider the following currents:

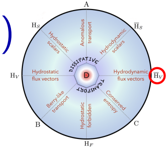
$$(T^{\mu\nu})_{\mathsf{B}} \propto \left(\mathcal{N}^{(\mu\nu)(\alpha\beta)} - \mathcal{N}^{(\alpha\beta)(\mu\nu)}\right) \pounds_{\boldsymbol{\beta}} \, g_{\alpha\beta}$$
$$(\mathcal{G}^\sigma)_{\mathsf{B}} = 0$$

  - Trivially solve adiabaticity equation
  - Manifestly **non-hydrostatic** ($\pounds_{\boldsymbol{\beta}} \, g_{\mu\nu} = 0$ in hydrostatics)
  - Seemingly not captured by Lagrangians (Class L)

- Easy task at any order in ∇:
  find all tensor structures $\mathcal{N}^{(\alpha\beta)(\mu\nu)}$ built out of $\{\boldsymbol{\beta}^\mu, g_{\mu\nu}\}$

- Examples in $d = 2 + 1$: Hall conductivity, Hall viscosity

Transverse non-hydrostatic free energy (Class $\overline{\mathrm{H}}_V$)

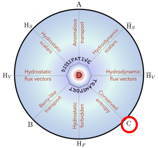


- Remember splitting: $\mathcal{G}^\sigma = \mathcal{S}\,\boldsymbol{\beta}^\sigma + \mathcal{V}^\sigma$ with $\boldsymbol{\beta}_\sigma \mathcal{V}^\sigma = 0$
- Consider solutions to adiabaticity equation with non-trivial and **non-hydrostatic** $\mathcal{V}^\sigma$
  - Transport genuinely due to free energy flux
- These are in general parameterized as

$$(T^{\mu\nu})_{\overline{\mathrm{H}}_V} \propto D_\rho \mathfrak{C}_{\mathcal{N}}^{\rho(\mu\nu)(\alpha\beta)}\, \pounds_{\boldsymbol{\beta}}\, g_{\alpha\beta} + 2\, \mathfrak{C}_{\mathcal{N}}^{\rho(\mu\nu)(\alpha\beta)}\, D_\rho \pounds_{\boldsymbol{\beta}}\, g_{\alpha\beta}$$

  - Easy task at any order in ∇:
    find all tensor structures $\mathfrak{C}_{\mathcal{N}}^{\rho(\mu\nu)(\alpha\beta)}$ built out of $\{\boldsymbol{\beta}^\mu, g_{\mu\nu}\}$

Conserved entropy current (Class C)



- Another trivial solution to adiabaticity equation: **exactly conserved entropy current**

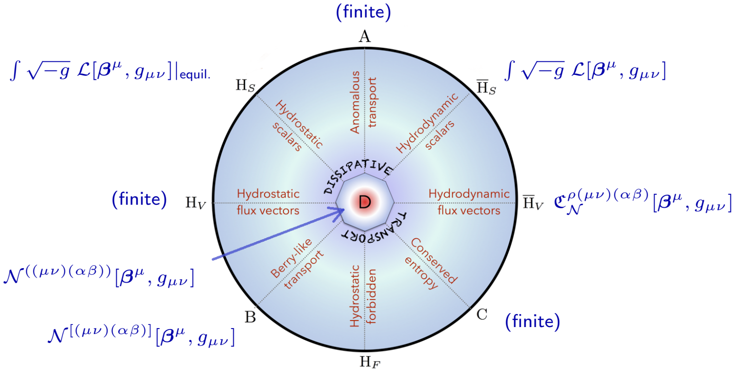
$$(J_S^\mu)_C = \mathsf{J}^\mu \quad \text{with} \quad D_\mu \mathsf{J}^\mu \equiv 0\,, \qquad (T^{\mu\nu})_C = 0$$

- If cohomologically non-trivial: describes **topological states** in the fluid (no energy/charge transport)
  - Example: Euler current $\mathsf{J}^\sigma_{\text{Euler}}$ in $d = 2+1$

Golkar-Roberts-Son '14

$$D_\sigma \mathsf{J}^\sigma_{\text{Euler}} \equiv 0\,, \qquad \int_{\Sigma_{\mathcal{M}}} \sqrt{-\gamma}\,(\mathsf{J}^\sigma_{\text{Euler}}\, u_\sigma) \quad \propto \quad \chi(\Sigma_{\mathcal{M}})$$

Summary: Classification of hydrodynamic transport



FH-Loganayagam-Rangamani '14 '15

Theorem: The eightfold way of hydrodynamic transport

- There are eight classes of $\{T^{\mu\nu}, J_S^\mu\}$ consistent with $\nabla_\mu J_S^\mu \gtrsim 0$.
- All of them can be constructed easily at all orders in ∇_μ.
- Constitutive relations not produced by this algorithm, are forbidden by second law (Class H_F).

Outline

- The hydrodynamic gradient expansion
- Classification of transport
- **Example: conformal fluid**
- Schwinger-Keldysh and emergent gauge symmetry
- Outlook and conclusion

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

- Most general 2nd order (neutral, Weyl-invariant) stress tensor:

$$
\begin{aligned}
T^{\mu\nu}_{(2)} = {} & (\lambda_1 - \kappa)\, \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} \\
& + (\lambda_2 + 2\tau - 2\kappa)\, \sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>} \\
& + \tau\, \left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) \\
& + \lambda_3\, \omega^{<\mu\alpha}\omega_\alpha{}^{\nu>} \\
& + \kappa\, \left(C^{\mu\alpha\nu\beta}u_\alpha u_\beta + \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} + 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right)
\end{aligned}
$$

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

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& + \tau \left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \rightarrow \text{Class } \overline{\mathrm{H}}_S \\
& + \lambda_3\, \omega^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \rightarrow \text{Class } \mathrm{H}_S \\
& + \kappa \left(C^{\mu\alpha\nu\beta} u_\alpha u_\beta + \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} + 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \rightarrow \text{Class } \mathrm{H}_S
\end{aligned}
$$

τ, λ_3, κ

Are all derivable from a Lagrangian (Class L)

$$\mathcal{L}_2^{\mathcal{W}} = \frac{1}{4}\left[-\frac{2\,\kappa}{(d-2)}({}^{\mathcal{W}}R) + 2\,(\kappa - \tau)\,\sigma^2 + (\lambda_3 - \kappa)\,\omega^2\right]$$

Note: λ_3 and κ are hydrostatic, τ is genuinely hydrodynamic

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

- Most general 2nd order (neutral, Weyl-invariant) stress tensor:

$$
\begin{aligned}
T^{\mu\nu}_{(2)} = & \, (\lambda_1 - \kappa)\, \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} && \to \text{Class D} \\
& + (\lambda_2 + 2\tau - 2\kappa)\, \sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \\
& + \tau\, \left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \to \text{Class } \overline{\mathrm{H}}_S \\
& + \lambda_3\, \omega^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \to \text{Class } \mathrm{H}_S \\
& + \kappa\, \left(C^{\mu\alpha\nu\beta}u_\alpha u_\beta + \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} + 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \to \text{Class } \mathrm{H}_S
\end{aligned}
$$

$(\lambda_1 - \kappa)$

Leads to entropy production $\Delta \simeq -(\lambda_1 - \kappa)\,\frac{1}{T}\,\sigma^\mu{}_\nu \sigma^\nu{}_\rho \sigma^\rho{}_\mu$

$\Rightarrow$ Dissipative ($\Rightarrow$ not captured by L-G Lagrangian)

(but unconstrained by second law, since $\sigma^3 \ll \sigma^2$)

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

- Most general 2[nd] order (neutral, Weyl-invariant) stress tensor:

$$
\begin{aligned}
T^{\mu\nu}_{(2)} = & (\lambda_1 - \kappa)\, \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} && \rightarrow \text{Class D} \\
& + (\lambda_2 + 2\tau - 2\kappa)\, \sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \rightarrow \text{Class B} \\
& + \tau\, \left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \rightarrow \text{Class } \overline{\mathrm{H}}_S \\
& + \lambda_3\, \omega^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \rightarrow \text{Class } \mathrm{H}_S \\
& + \kappa\, \left(C^{\mu\alpha\nu\beta}u_\alpha u_\beta + \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} + 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \rightarrow \text{Class } \mathrm{H}_S
\end{aligned}
$$

$(\lambda_2 + 2\tau - 2\kappa)$

Is of the form of a Class B constitutive relation

$$
(T^{\mu\nu})_{\mathrm{B}} \equiv -\frac{1}{4}\left(\mathcal{N}^{(\mu\nu)(\alpha\beta)} - \mathcal{N}^{(\alpha\beta)(\mu\nu)}\right)\, \pounds_{\boldsymbol{\beta}}\, g_{\alpha\beta}
$$
$$
(\mathcal{G}^\sigma)_{\mathrm{B}} = 0
$$

because of orthogonality: $\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\, \pounds_{\boldsymbol{\beta}}\, g_{\mu\nu} = 0$

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

- Out of 5 transport coefficients, 3 come from a Lagrangian: τ, λ_3 and κ
- For fluids described by $\mathcal{L}[\boldsymbol{\beta}^\mu, g_{\mu\nu}]$, the other 2 combinations are zero:

$$(\lambda_1 - \kappa) = 0 \quad \text{and} \quad (\lambda_2 + 2\tau - 2\kappa) = 0$$

  - These relations have been **observed in Einstein gravity** *Haack-Yarom '08*
    - Our simple Lagrangians seem to know about holography
    - Derive $\mathcal{L}[\boldsymbol{\beta}^\mu, g_{\mu\nu}]$ from gravity directly? *Nickel-Son '10*

      de Boer et al. '15, Crossley et al. '15
  - First relation ensures **no entropy production at subleading order** (this is not required by second law!)
    $\rightarrow$ **"Principle of minimum dissipation"** in holography?

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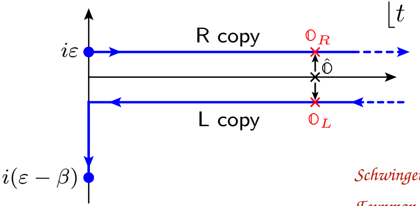
Outline

- The hydrodynamic gradient expansion
- Classification of transport
- Example: conformal fluid
- **Schwinger-Keldysh and emergent gauge symmetry**
- Outlook and conclusion

Schwinger-Keldysh doubling

- Non-equilibrium effective field theory in general described by **Schwinger-Keldysh** formalism
  - Systems in mixed state: path integral evolves both $|\cdot\rangle$ and $\langle\cdot|$

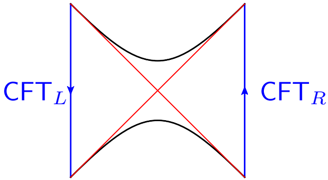


- **Double fields & symmetries:**

$$\mathcal{H}_{phys} \subset \mathcal{H}_R \otimes \mathcal{H}_L$$

Schwinger, Keldysh, Feynman-Vernon, '60s

- In principle many applications: off-equilibrium physics, dissipation, gravity with horizons, complementarity, ...



Schwinger-Keldysh hydrodynamics?

- Can we upgrade Class L Lagrangians to describe all 8 classes, using Schwinger-Keldysh?
- Would like to double the sources: $g_{\mu\nu} \to \{g_{\mu\nu}, \tilde{g}_{\mu\nu}\}$
  - ...and write something like

$$S_{\text{eff}} = \int \sqrt{-g}\, \frac{1}{2}\, T^{\mu\nu}[\boldsymbol{\beta}^\mu, g_{\mu\nu}]\, \tilde{g}_{\mu\nu}$$

$$\Rightarrow \quad \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{eff}}}{\delta \tilde{g}_{\mu\nu}} = T^{\mu\nu}$$

- This would allow us to get **any** $T^{\mu\nu}$ from an action. Including Class H_F !

Problem:

Just blindly doubling gives too much freedom.
Can easily violate 2nd law.

  - Doubling is very powerful. But how to control it?
  - Important obstacle for systematic understanding of eff. field theory of mixed states (dissipation, entanglement, horizons, ...)

A new macroscopic symmetry

We found a consistent **Schwinger-Keldysh doubling for hydrodynamics.**
It is constrained in exactly the right way to reproduce the eightfold way.
"Danger of doubling" controlled by introducing an **emergent** $U(1)_{\mathsf{T}}$ **symmetry**.

A new macroscopic symmetry

We found a consistent **Schwinger-Keldysh doubling for hydrodynamics.**
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"Danger of doubling" controlled by introducing an **emergent** $U(1)_\mathsf{T}$ **symmetry**.

- Remember Class L free energy current: $\mathcal{G}^\mu = -\mathcal{L}\, u^\mu + T(\text{bdy. term}_{\boldsymbol{\beta}})^\mu$
  - $\mathrm{N}^\mu \equiv -\frac{\mathcal{G}^\mu}{T}$ is **Noether current for diffeos along** $\boldsymbol{\beta}^\mu$
- Postulate a $U(1)_\mathsf{T}$ gauge symmetry with gauge field $\mathsf{A}^{(\mathsf{T})}{}_\mu$ coupling to N^μ

Proposed field content:

- Hydrodynamic field: $\boldsymbol{\beta}^\mu$
- Background source: $g_{\mu\nu}$
- SK copy of source: $\tilde{g}_{\mu\nu}$
- $U(1)_\mathsf{T}$ gauge field: $\mathsf{A}^{(\mathsf{T})}{}_\mu$

The eightfold master Lagrangian (Class $\mathrm{L_T}$)

- **Any constitutive relations** $\{T^{\mu\nu}, \mathcal{G}^\sigma\}$ **which satisfy adiabaticity equation** can be obtained from a diffeo and $U(1)_\mathsf{T}$ invariant Lagrangian:

$$\mathcal{L}_\mathsf{T} = \frac{1}{2} T^{\mu\nu}[\boldsymbol{\beta}^\mu, g_{\mu\nu}]\tilde{g}_{\mu\nu} - \frac{\mathcal{G}^\sigma[\boldsymbol{\beta}^\mu, g_{\mu\nu}]}{T} \mathsf{A}^{(\mathsf{T})}{}_\sigma$$

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  - Bianchi identity for $U(1)_\mathsf{T}$ invariance reduces to adiabaticity equation
  - Equations of motion are:
    - As in Class L: $D_\nu T^{\mu\nu} \simeq 0$
    - From $\mathsf{A}^{(\mathsf{T})}{}_\sigma$: $D_\mu J^\mu_S \simeq 0$

- Conversely: any diffeo and $U(1)_\mathsf{T}$ invariant Lagrangian gives adiabatic constitutive relations
  - That is: $U(1)_\mathsf{T}$ **invariance gives precisely the right constraint!**

Outline

- The hydrodynamic gradient expansion
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- Schwinger-Keldysh and emergent gauge symmetry
- **Outlook and conclusion**

Outlook: Talk next Friday (Nov 20, string group meeting)

- Try to derive this scenario from first principles
- Study Schwinger-Keldysh path integrals in detail
- Find very interesting and universal structures:
  - Hidden **topological (BRST) supersymmetry** behind every SK path integral (incl. relativistic fluids)
  - Associated **ghosts** are crucial to retain unitarity
  - **Gauge theory of entropy:** general field theory argument
- Formulate eff. action for **all 8 classes**
  - Dissipation has to do with ghosts
  - Via AdS/CFT: all these features should play some role in black holes $\rightarrow$ lots of interesting conjectures to work on

Summary

- 8 classes of constitutive relations consistent with the Second Law
- I gave a recipe for constructing them at any order
- The classification is useful and physical:
  - Computations become simpler in this framework
    (classification often tells what is the "nicest" basis to work in)
  - Conjecture: long-wavelength **near-horizon AdS dynamics** can be usefully characterized using the Eightfold Way
  - **"Minimum dissipation conjecture":**
    Holographic fluids optimize entropy production.
- Schwinger-Keldysh doubling is powerful but dangerous
  - Proposal: **emergent** $U(1)_{\mathsf{T}}$ **gauge symmetry** controls the doubling in hydrodynamics
  - Next week: systematic justification of $U(1)_{\mathsf{T}}$