General Relativity

Physics Honours 2006

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Lecture Notes 7

Black Holes

We have seen previously that something weird happens when we fall towards the origin of the Schwarzschild metric; while the proper motion for the fall is finite, the coordinate time tends to $r=2M$ as $t \to \infty$.

Clearly, there is something weird about $r=2M$ (the Schwarzschild radius) for massive particles. But what about light rays? Remembering that light paths are **null** so we can calculate structure of radial light paths.

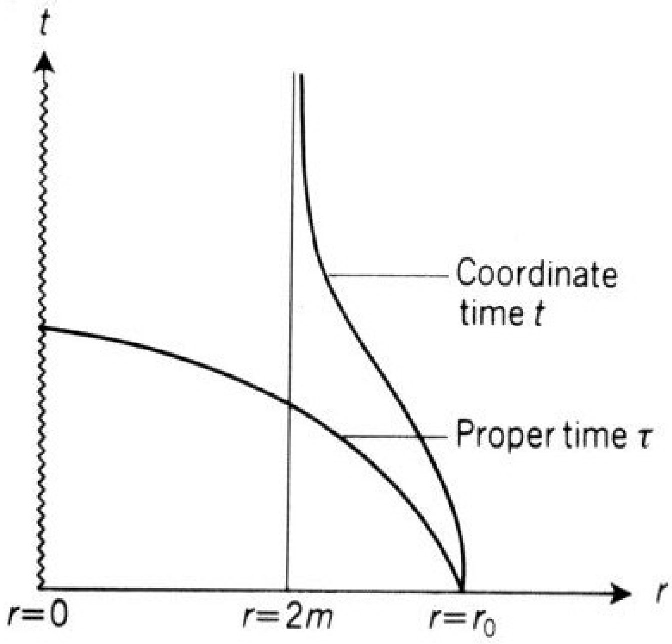
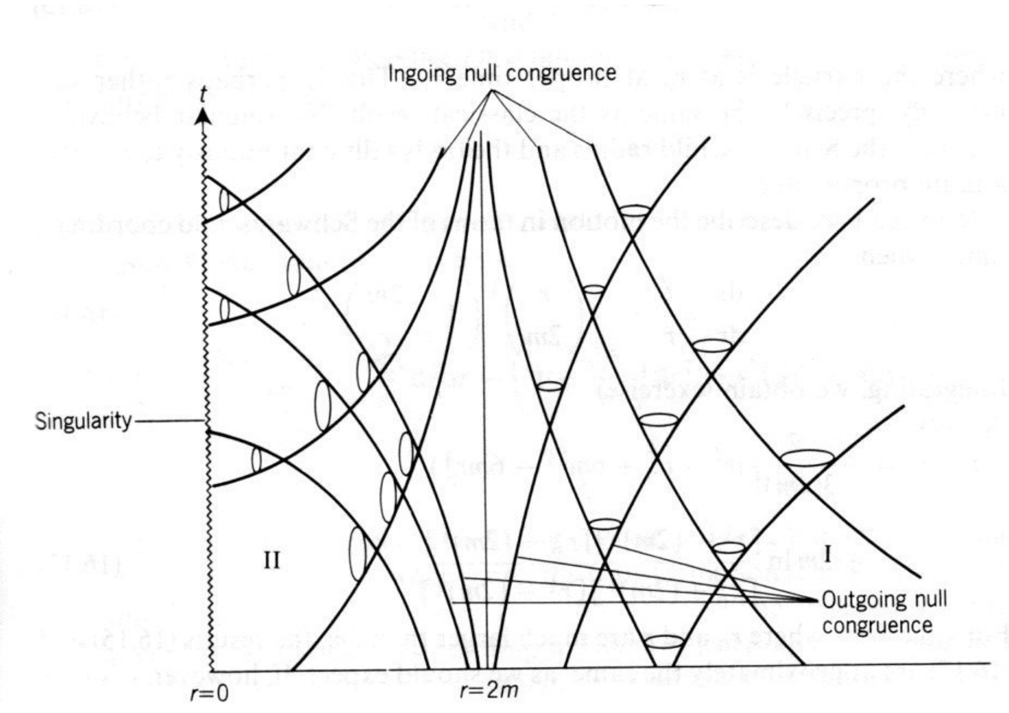


Fig. 16.8 Radially infalling particle in times τ and t.

Black Holes



We can calculate the gradients of light rays from the metric

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}$$

Again, light curves tell us about the future of massive particles.

Clearly the light cones are distorted, and within *r=2M* all massive particles are destined to hit the origin (the central singularity). In fact, once within this radius, a massive particle will not escape and is trapped (and doomed)!

But how do we cross from inside to outside?

Eddington-Finkelstein

While Eddington figured out the solution in the early 1900s, but it was Finkelstein in the late 1950s who rediscovered the answer. Basically, we will just make a change of coordinates;

$$t = v - r - 2M log \left| \frac{r}{2M} - 1 \right|$$

And the Schwarzschild metric can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

The geometry is the same, but the metric now contains off-axis components. But notice now that the metric does nothing weird at *r=2M* (but still blows up at *r=0*).

Eddington-Finkelstein

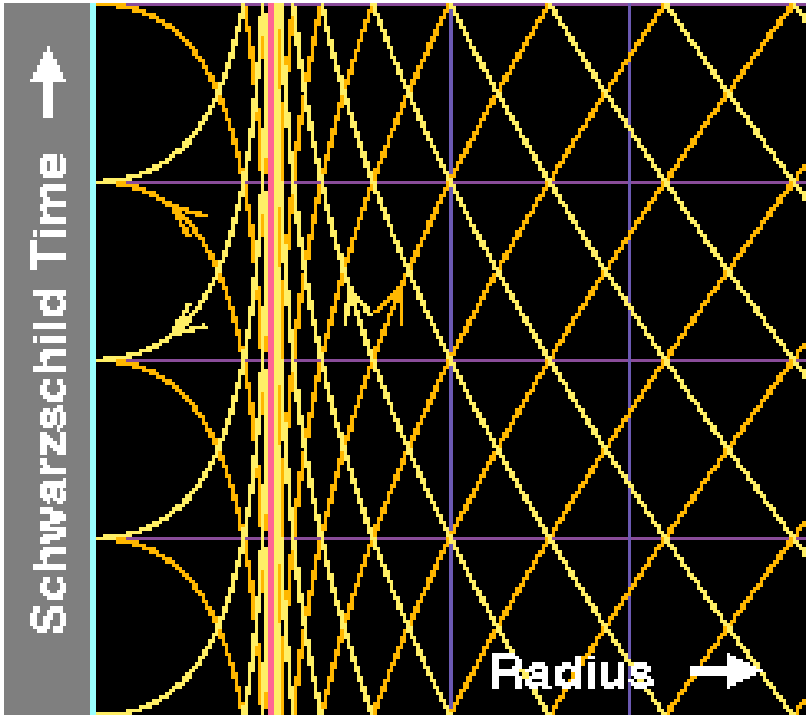
Light rays are still null, so we can find the light cones

$$-\left(1-\frac{2M}{r}\right)dv^2+2dvdr=0$$

Clearly, *v=const* represents a null geodesic (and these are ingoing rays). There is another that can be integrated;

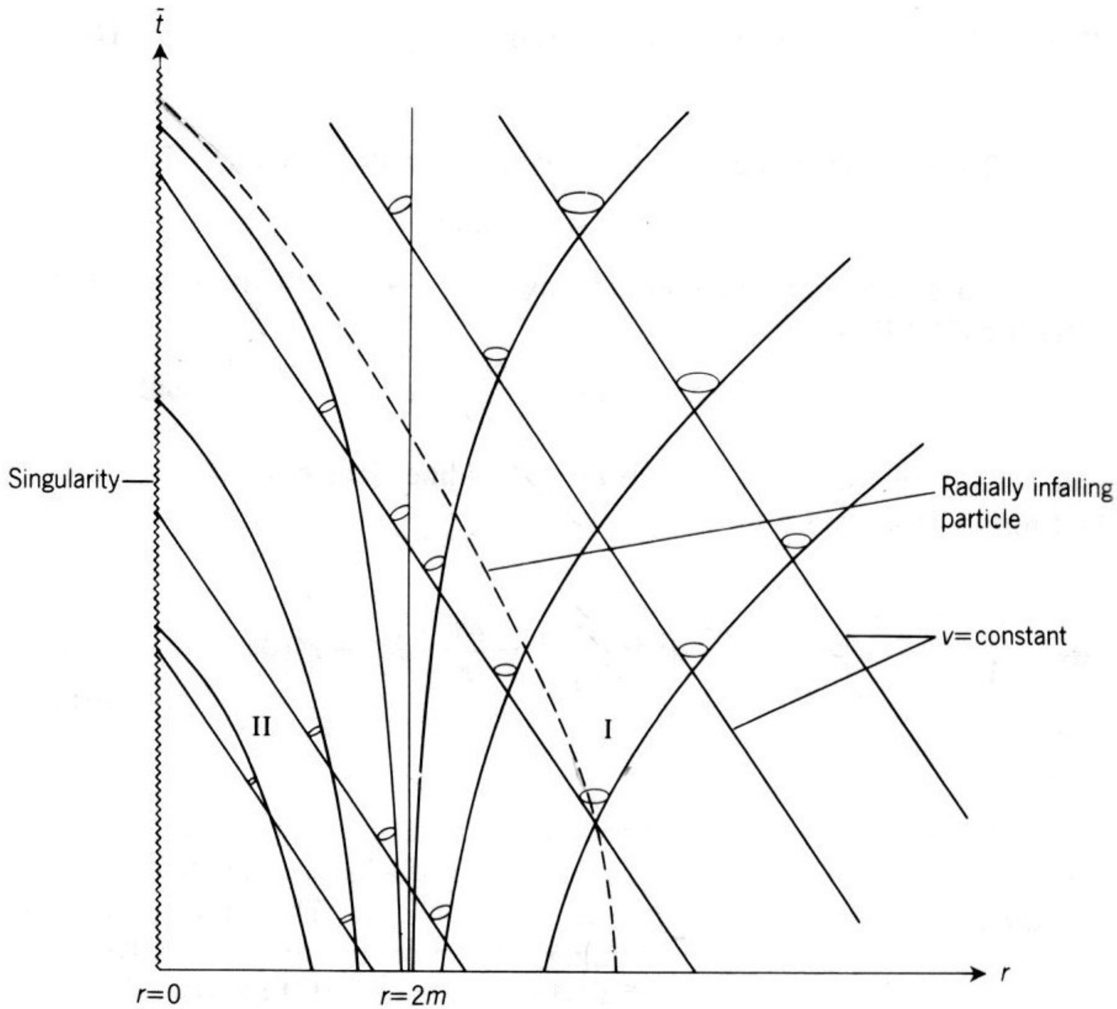
$$v-2\left(r+2Mlog\left|\frac{r}{2M}-1\right|\right)=const$$

So, the change in coordinates removes the coordinate singularity and now light (and massive particles) happily fall in.



http://casa.colorado.edu/~ajsh/

Eddington-Finkelstein



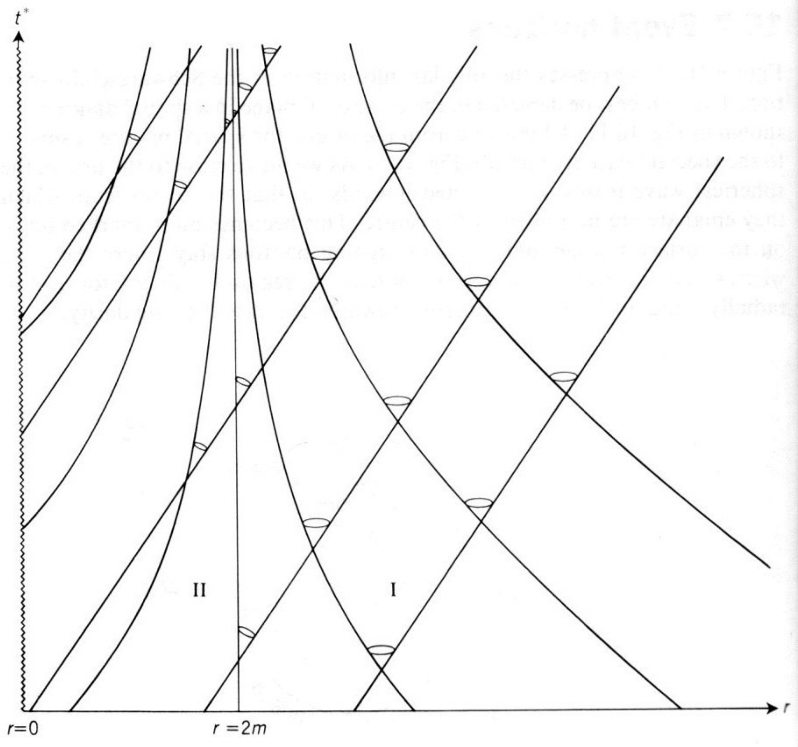
To create this figure, we define a new time coordinate of the form;

$$\tilde{t} = v - r$$

This straightens in-going light rays (making it look like flat space time).

However, outgoing light rays are still distorted.

White Holes



Another coordinate transformation can straighten outgoing light rays.

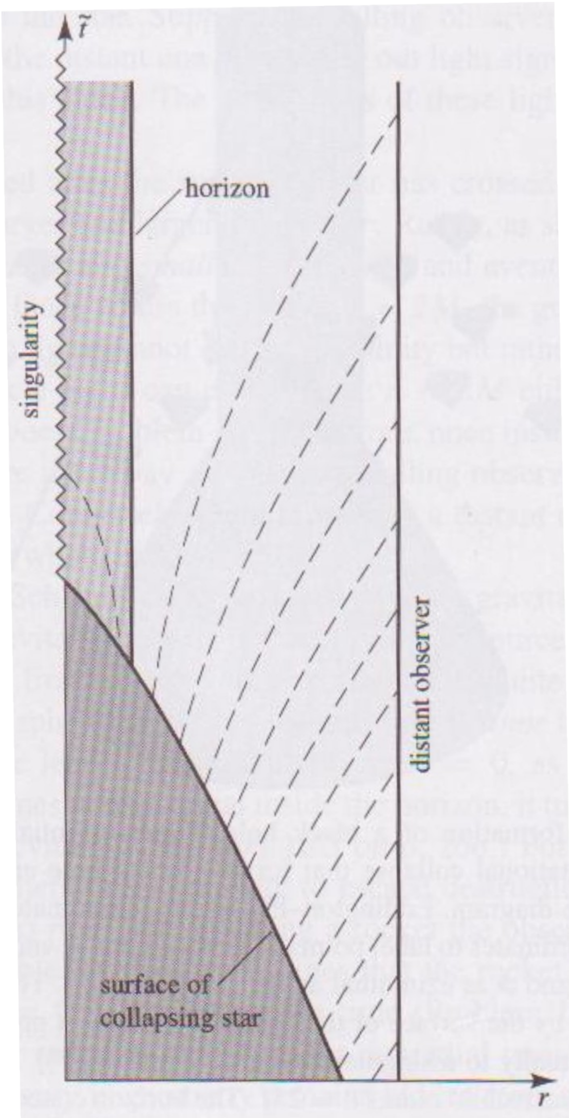
The result is a **white hole** and massive particle at *r<2M* are destined to be ejected and cannot return.

Note that while this is still the Schwarzschild solution, this behaviour is not seen in the original solution.

$$t = w + r + 2M \log \left| \frac{r}{2M} - 1 \right|$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dw^2 - 2dwdr + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Collapse to a Black Hole



Consider the collapse of a pressureless (dust) star. The surface of the will collapse along time-like geodesics. We know that the proper time taken for a point to collapse to *r=0* is

$$r(\tau) = (3/2)^{\frac{2}{3}}(2M)^{\frac{1}{3}}(\tau_* - \tau)^{\frac{2}{3}}$$

While the Schwarzschild metric blows up at the horizon, the E-F coordinates remain finite and you can cross the horizon.

Once across the horizon, the time to the origin is

$$\Delta\tau = \frac{4}{3}M$$

Which is 10^{-5}*s* for the Sun.

Collapse to a Black Hole

Remember that in E-F coordinates, outgoing light rays move along geodesics where

$$v - 2\left(r + 2M log\left|\frac{r}{2M} - 1\right|\right) = const$$

Let's consider an emitter at *(v_E,r_E)* and receiver at *(v_R,r_R)*. If the receiver is at large distances, then the *log* term can be neglected. When the emitter is close to *r=2M* the *log* term dominates. For the distant observer then

$$v \sim t_R + r_R \quad (book\ \ typo)$$

Where *t* is the Schwarzschild time coordinate (which will equal the proper time of the observer as the spacetime far from the hole is flat).

Collapse to a Black Hole

Keeping the dominant terms, the result is

$$\frac{r_E}{2M} - 1 \sim e^{-\frac{t_R - r_R}{4M}}$$

As $r \to r_E$, $t_R \to \infty$. Photons fired off at regular intervals of proper time are received later and later by the observer.

These photons are also redshifted. If the photons are emitted at regular intervals of $\Delta\tau$ (proper time) then

$$\Delta r_E = u^r \Delta\tau$$

$$-\frac{|u^r|\,\Delta\tau}{2M} = \frac{\Delta r_E}{2M} \sim -\frac{\Delta t_R}{4M} e^{-(t_R - r_R)/4M}$$

Collapse to a Black Hole

Remembering that the frequency of the emitted and received photons are inversely related to the ratio of the emitted and received time between photons, then

$$\omega_R(t_R) \propto \omega_* e^{-\frac{t_R}{4M}}$$

So the distant observer sees the surface of the star collapsing, but as it approaches $r=2M$ it appears to slow and the received photons become more and more redshifted.

The distant observer never sees the surface cross the Schwarzschild radius (or horizon) although we see from the E-F coordinates, the mass happily falls through.

Kruskal-Szekeres

We can take the game of changing coordinates even further with Kruskal-Szekeres coordinates. Starting again with the Schwarzschild metric, and keeping the angular coordinates unchanged, the new coordinates are given by

$$\begin{aligned} U &= \sqrt{\left|1-\frac{r}{2M}\right|}\; e^{r/4M}\; c/s\,(t/4M) \\ V &= \sqrt{\left|1-\frac{r}{2M}\right|}\; e^{r/4M}\; s/c\,(t/4M) \end{aligned}$$

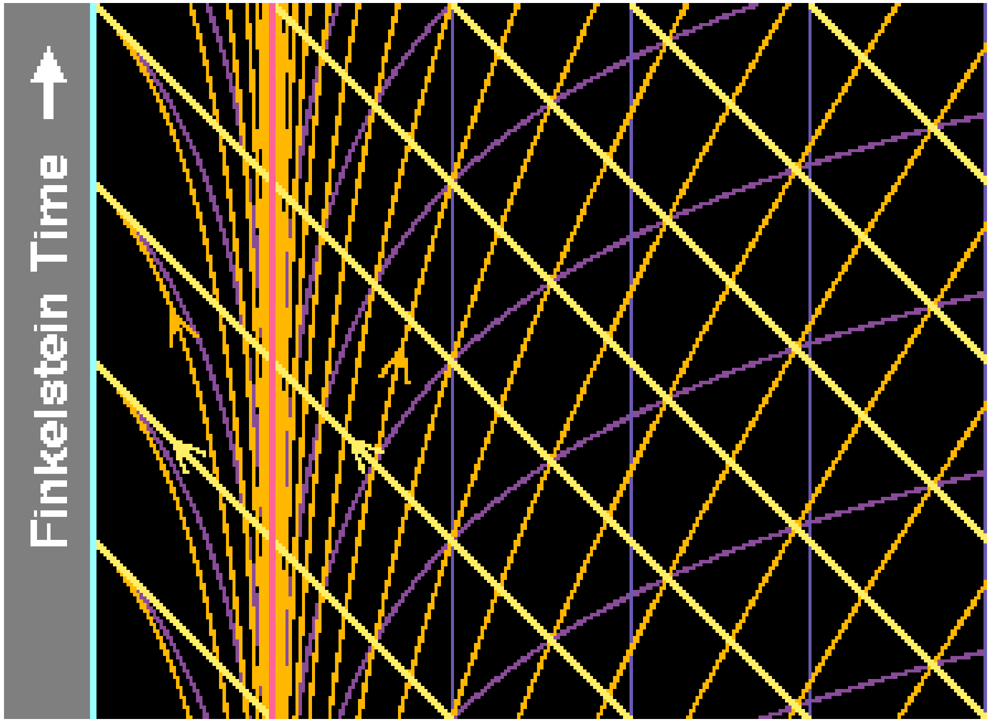
Where *c* & *s* are *cosh* & *sinh* with the first combination used for *r>2M* and the second for *r<2M*. We also find that

$$U^2 - V^2 = \left(\frac{r}{2M} - 1\right)\; e^{r/2M}$$

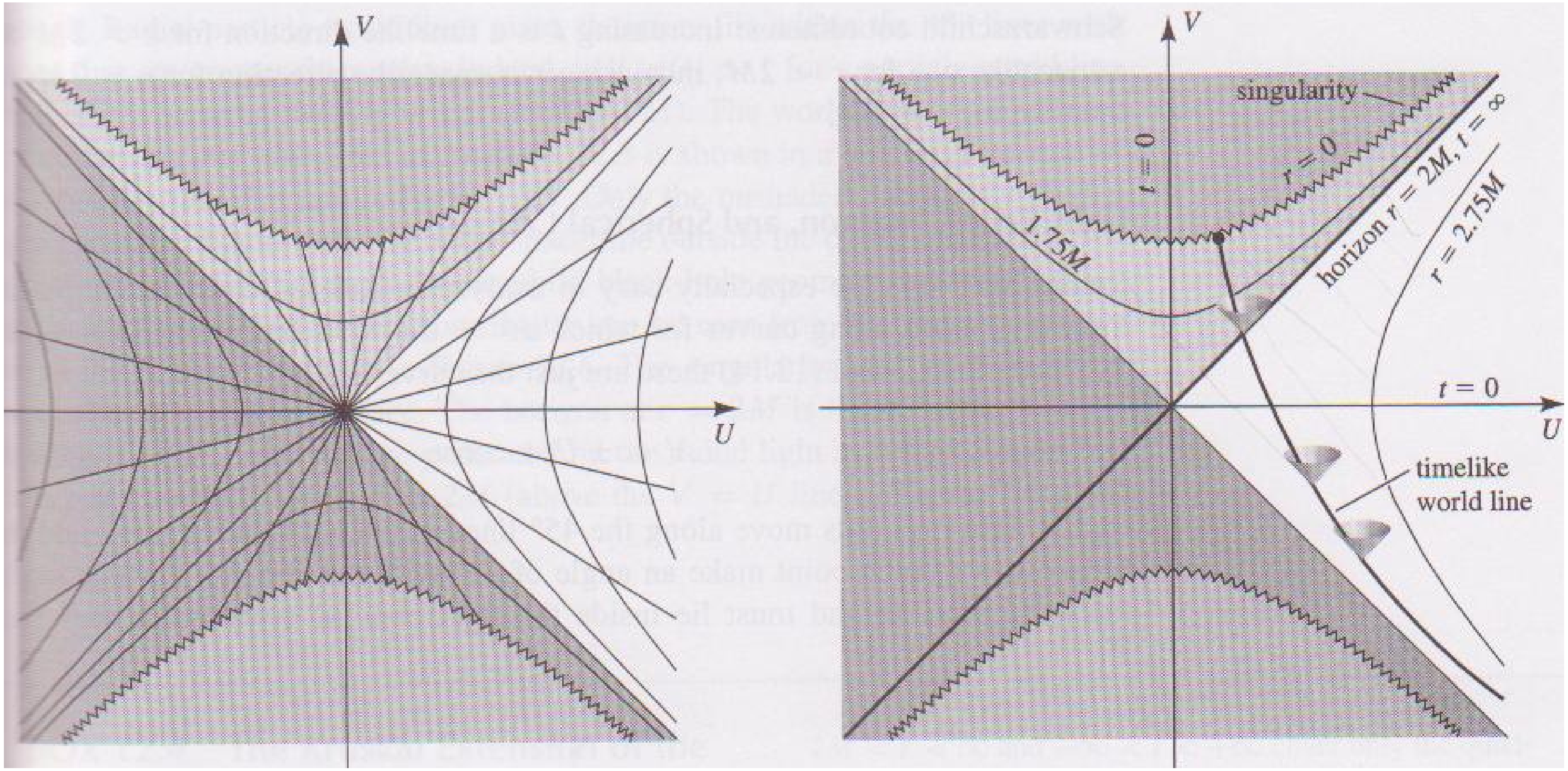
Kruskal-Szekeres

The resultant metric is of the form

$$ds^2 = \frac{32M^3}{r}e^{-r/2M}\left(dU^2 - dV^2\right)+r^2(d\theta^2+sin^2\theta d\phi^2)$$



Kruskal-Szekeres

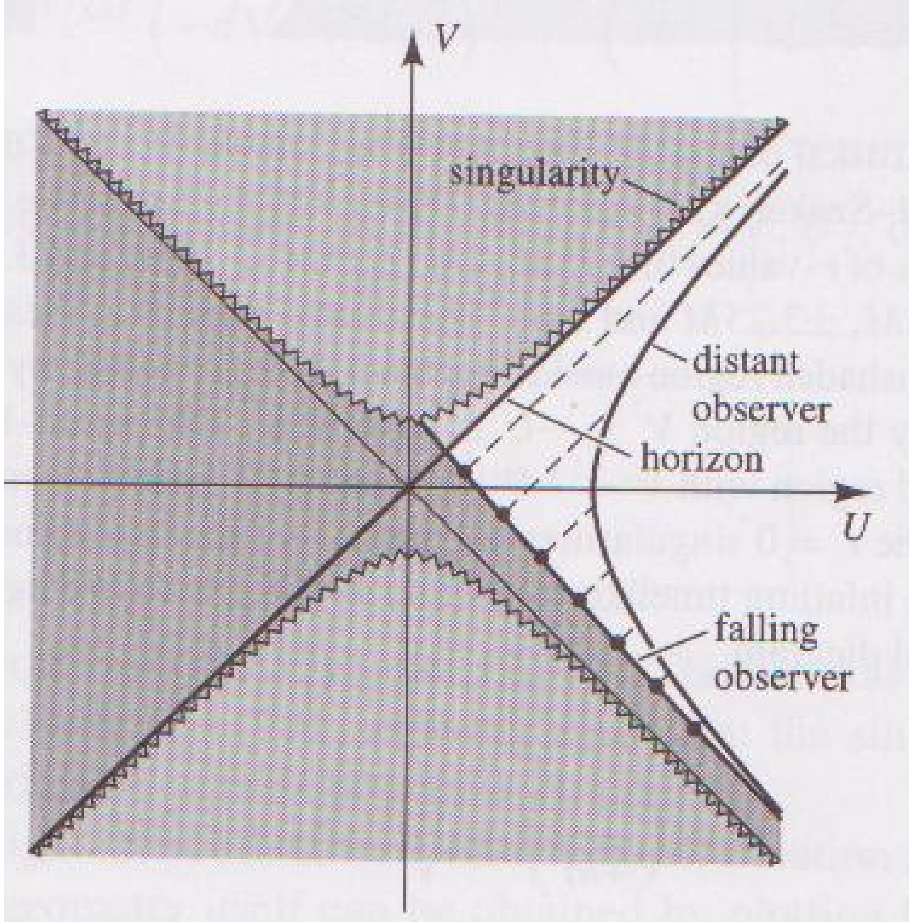


Lines of constant *t* are straight, while those at constant *r* are curves. Light cones are at 45°, as in flat spacetime. We have our universe, plus a future singularity (black hole) and past singularity (white hole). There also appears to be another universe over to the left.

Kruskal-Szekeres

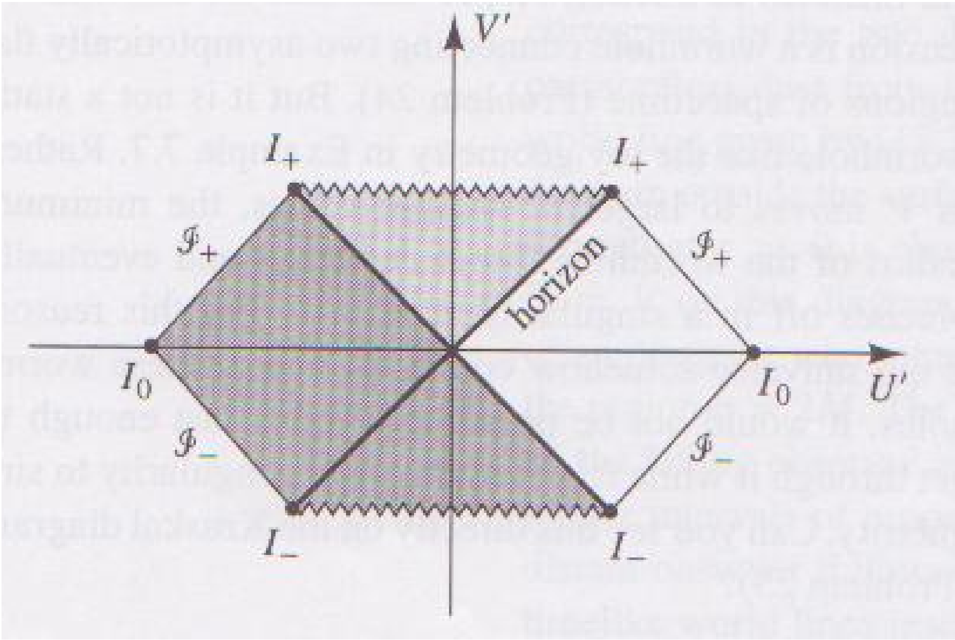
We can examine the radial infall of matter in these coordinates. The distant observer moves along a line of constant radius, while the matter falls in emitting photons. Again, the distant observer sees the photons arriving at larger intervals and never sees the matter cross the horizon.

Note that once inside the horizon, the matter **must** hit the central singularity.



Penrose

Penrose mapped the Kruskal coordinates further, such that now we get two entire universes on a single page. This is an example of **maximal compactification**.



$$U = (v - u)/2 \qquad u' = tan^{-1}(u) = V' - U'$$

$$V = (v + u)/2 \qquad v' = tan^{-1}(v) = V' + U'$$