On the time-reversal anomaly of 2+1d TQFTs

Yuji Tachikawa (Kavli IPMU)

in collaboration with
Kazuya Yonekura (Kavli IPMU)

Natifest, September 2016

It is a great honor to speak at Natifest.

I first spoke with Nati when he offered me a postdoc position by phone.

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I got a high fever right after that, and was in bed for a few days.

That was 11 years ago. Time flies!

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and Nati’s office door was often open.

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For example, one cold winter day,
I overheard Davide Gaiotto chatting with Nati.

As they sounded very excited, I asked them if I could join.

It turned out that Davide was explaining to Nati
what became known as the class S theory!

It was a few months before the first paper came out,
and it gave me a head start working in this business.

Another episode:

In 2008, I gave a local seminar here,
on a counterexample to the a-theorem I thought I found with Al Shapere.

Nati didn't like it.

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Nati didn't like it. At all.

Two years later, he told me he debugged it with Davide.

In the end it became a paper by Nati, Davide and me.

Soon the a-theorem was proved by Zohar and Adam Schwimmer.

So far I have three papers with Nati,
and I learned a lot by working with him.

Of course I learned a lot about physics,
but somehow I feel I learned more about the attitude toward physics.

For example:

- the importance of **finding the right question to ask,**
- of **identifying the crucial elements in the answer,**
- and **how to concisely express those elements in a paper.**

When he edits the draft, it often becomes **shorter** and **clearer**.

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I remain such a loyal follower of Nati,

that when I heared a rumor last summer that
he and Edward were working on topological phases,

I decided I should work on it too!

Today I'd like to say something about it.

All is based on my collaboration with **Kazuya Yonekura**,
a postdoc at IPMU and a former postdoc here at IAS.

Today I'd like to discuss

Anomaly of time-reversal symmetry of 2+1d systems

- What is it?
- What are some systems that have it?
- How should one determine it?

I'd like to first remind ourselves of a completely understood case of:

Anomaly of $\mathbf{U(1)}$ symmetry of 3+1d systems

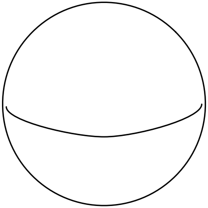
- What is it?
- What are some systems that have it?
- How should one determine it?

Anomaly of $\mathbf{U}(1)$ symmetry of 3+1d systems:

What is it?

Phase ambiguity of the partition function
in the presence of background $\mathbf{U}(1)$ gauge field.

The phase ambiguity occurs in a controlled way, as follows.

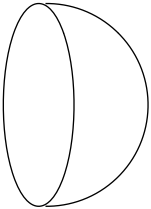


Consider 5d closed manifold $\boldsymbol{X}$ with a background $\mathbf{U(1)}$ field, with the CS action

$$\exp\left[2\pi i k \int_X A \wedge F \wedge F\right]$$

This is invariant under the gauge transformation if $k \in \mathbb{Z}$.

(I would be sloppy about the normalizations. Forgive me.)



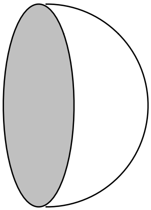
If the 5d manifold X has a boundary, $M = \partial X \neq \varnothing$, the CS action

$$\exp\left[2\pi i k \int_X A \wedge F \wedge F\right]$$

is **not invariant** even when $k \in \mathbb{Z}$,

due to the gauge variation at the boundary.

You can add **something physical** at the boundary $M = \partial X$



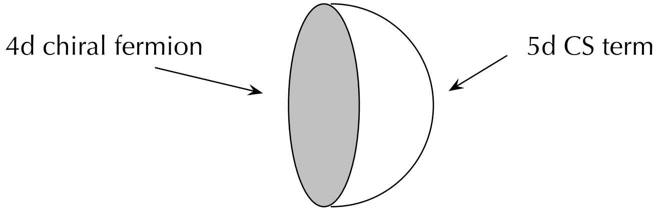
so that the combination

$$Z_M[A|_M] \exp\left[2\pi i k \int_X A \wedge F \wedge F\right]$$

is **invariant**: the phase ambiguities of two terms cancel each other.

This is also called the anomaly inflow. [Callan-Harvey, …]

A typical example of such **something physical** is, of course, charged chiral fermions on M.



Is there another way to see such chiral fermions arise on the boundary?

Given k charged massive **5d** fermion with the mass term

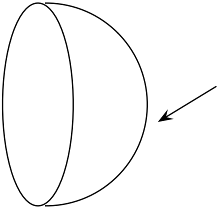
$$m\psi\bar{\psi},$$

integrating them out generates the CS term

$$\exp[\pm 2\pi i \frac{k}{2} \int A \wedge F \wedge F]$$

where the sign $\pm$ is the sign of m.

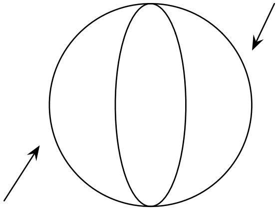
Now, instead of



$$\mathbf{exp}\left[2\pi i k \int_X A \wedge F \wedge F\right]$$

We can consider

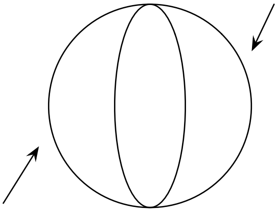
$$\exp[+2\pi i \frac{k}{2} \int A \wedge F \wedge F]$$



$$\exp[-2\pi i \frac{k}{2} \int A \wedge F \wedge F]$$

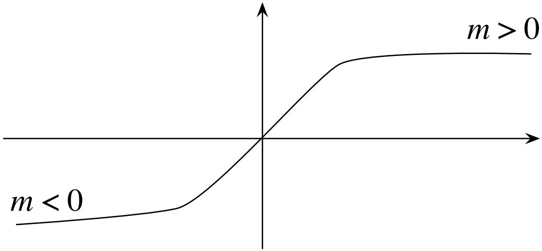
which you can represent as

k 5d fermions with mass $m > 0$

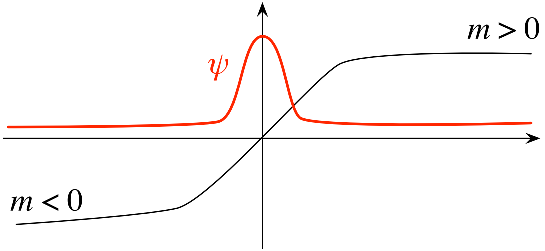


k 5d fermions with mass $m < 0$

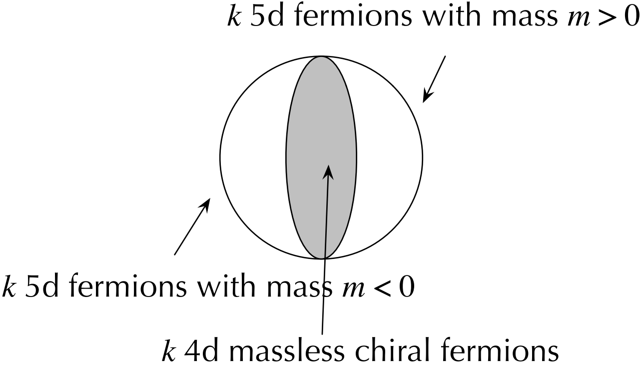
but when the fermion mass is space dependent



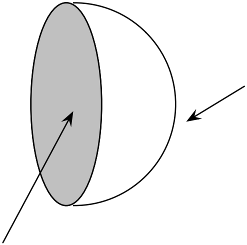
we know that there is a zero mode



so we have


k 5d fermions with mass $m > 0$
k 5d fermions with mass $m < 0$
k 4d massless chiral fermions

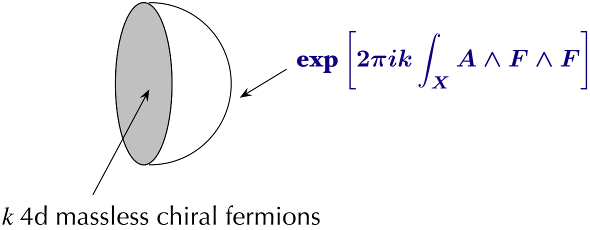
which is



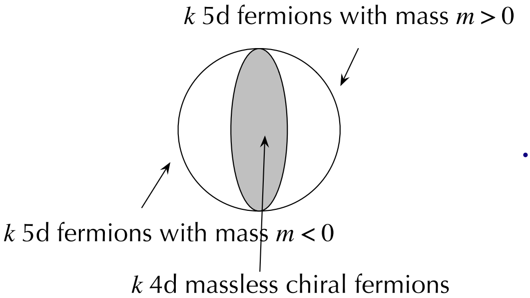
$$\exp\left[2\pi i k \int_X A \wedge F \wedge F\right]$$

k 4d massless chiral fermions

Note the equality of pictures



and



So far we recalled

Anomaly of $\mathbf{U(1)}$ symmetry of 3+1d systems.

But today I wanted to discuss

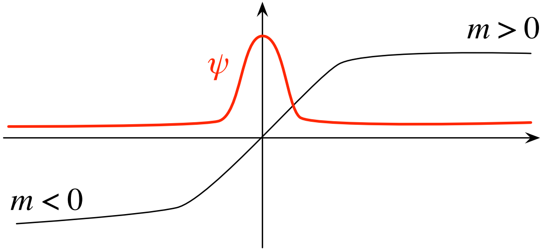
Anomaly of time-reversal symmetry of 2+1d systems.

A typical time-reversal invariant system in **3+1d** is a massive Majorana fermion with the mass term

$$m\psi\psi$$

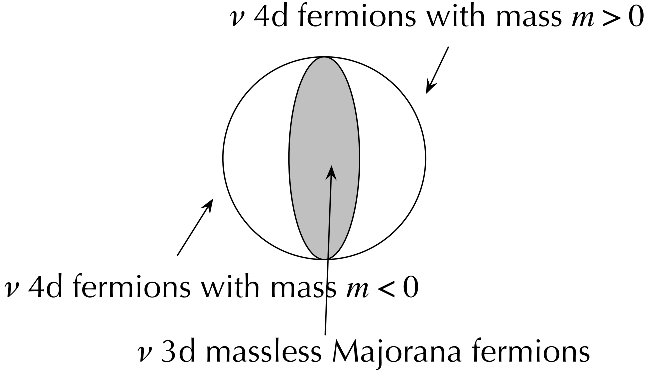
where $m \in \mathbb{R}$ to be invariant under the time reversal.

If we make m space-dependent, we have a zero-mode



which is a massless Majorana fermion in **2+1d**.

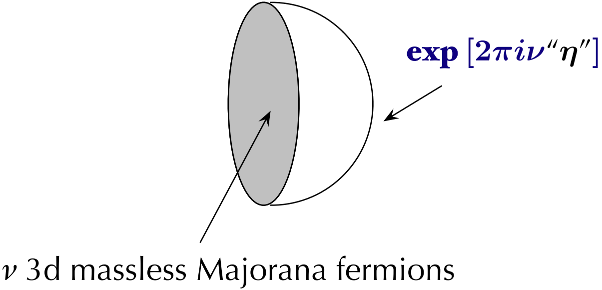
So we can have the situation



where we consider general **non-orientable manifolds**,

to give an **equivalent of 'background gauge field for time-reversal.'**

We can integrate out the massive fermions to find



where “η” is the so-called eta invariant,
that plays the role of the CS term in this case.

Condensed-matter theorists know the bulk by the name
3+1d topological superconductor.

On a closed (in general non-orientable) 4d manifold X,

$$\exp\left[2\pi i \text{“}\eta\text{''}\right]$$

is always a **16th root of unity**. Therefore, in the expression

$$\exp\left[2\pi i \nu \text{“}\eta\text{''}\right]$$

only $\nu \in \mathbb{Z}_{16}$ **matters.**

This is known under the name
$\mathbb{Z}_{16}$**-classification of the 3+1d topological superconductor.**

For us, this means that

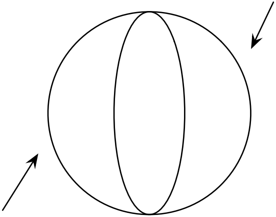
the time-reversal anomaly of 2+1d systems is a quantity $\boldsymbol{\nu} \in \mathbb{Z}_{\mathbf{16}}$.

For example, $\boldsymbol{\nu}$ massless 2+1d Majorana fermions have the value

$$\boldsymbol{\nu} \in \mathbb{Z}_{\mathbf{16}}.$$

What are other 2+1d systems with time-reversal anomalies?

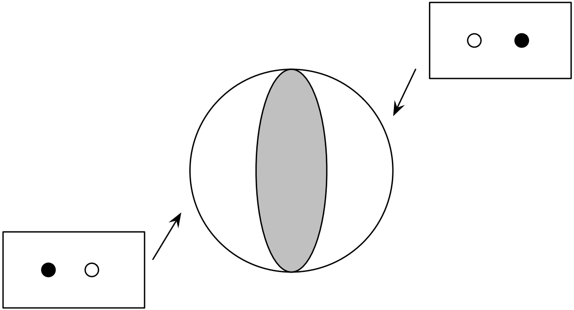
Let's start from



Regard **3** fermions as an adjoint of $\mathbf{SU(2)}$, and
couple dynamical $\mathbf{SU(2)}$ gauge field to it.

This is $\mathcal{N}=1$ $\mathbf{SU(2)}$ SYM softly broken by the gaugino mass.

So we have a domain wall between two vacua of $\mathcal{N}=1$ $\mathbf{SU(2)}$ SYM



on which it's known that we have

a 3d goldstino $+$ $\mathbf{U(1)_2}$ CS.

[Acharya-Vafa,...]

We started from $\boldsymbol{\nu = 3}$ 4d fermions, so

$$\boldsymbol{\nu}[\text{a 3d goldstino}] + \boldsymbol{\nu}[\mathbf{U(1)_2}\ \text{CS}] = \mathbf{3.}$$

Clearly

$$\boldsymbol{\nu}[\text{a 3d goldstino}] = \mathbf{1.}$$

Therefore

$$\boldsymbol{\nu}[\mathbf{U(1)_2}\ \text{CS}] = \mathbf{2.}$$

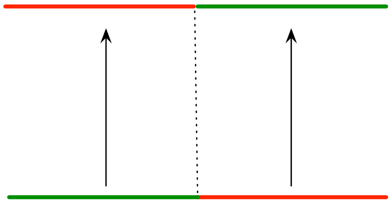
It is interesting that a theory without any massless things can have an anomaly. This can't happen for the anomaly of continuous symmetry.

So, suppose we're given a time-reversal symmetric 2+1d TQFT.

How do we determine its anomaly $\nu \in \mathbb{Z}_{16}$ directly?

[YT-Kazuya Yonekura, to appear soonish]

Consider putting the 3d theory on a crosscap:

$\mathbb{R}_{\boldsymbol{t}} \times$ 

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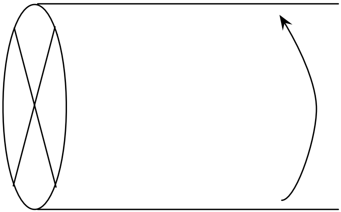
$\mathbb{R}_{\boldsymbol{t}} \times$ 

There is an isometry rotating it.

The associated conserved quantity is the momentum $\boldsymbol{p}$.

In a non-anomalous theory, we have

$$\boldsymbol{p = n \in \mathbb{Z}.}$$

This is because the $\boldsymbol{2\pi}$ rotation should not do anything:

$$\mathbf{exp}\,[\boldsymbol{2\pi i p}] = \mathbf{1.}$$

In an anomalous theory, this might not hold, because of phase ambiguity:

$$\exp [2\pi i p] \neq 1.$$

This phase ambiguity should be 'linear' in ν, so we have

$$\exp [2\pi i p] = \exp [2\pi i c\nu]$$

for some number c.

Equivalently,

$$p = n + c\nu, \qquad n \in \mathbb{Z}.$$

We can fix c by considering a system whose ν is known, for example a 3d Majorana fermion for which $\nu = 1$.

One finds by an explicit computation that $c = 1/16$. [Hsieh-Cho-Ryu,1503.01411]

Therefore:

$$p = n + \frac{\nu}{16}, \qquad n \in \mathbb{Z}.$$

The time-reversal anomaly manifests itself as the anomalous momentum on the crosscap.

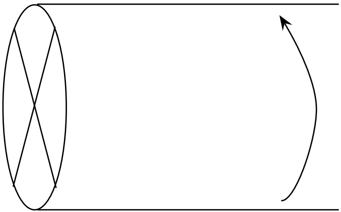
For 1+1d systems the same thing was pointed out in [Cho-Hsieh-Morimoto-Ryu, 1501.07285]

How do we compute the anomalous momentum for a 2+1d TQFT?

Suppose we have

$$p = \frac{\nu}{16} \mod \mathbb{Z}$$

on

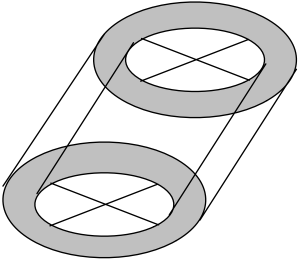


.

We have

$$T|\text{crosscap}\rangle = e^{2\pi i\nu/16}|\text{crosscap}\rangle$$

where $T \in \mathbf{SL}(2, \mathbb{Z})$ and $|\text{crosscap}\rangle$ is a state on T^2 generated by

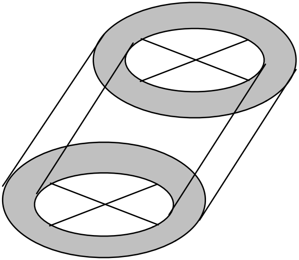


,

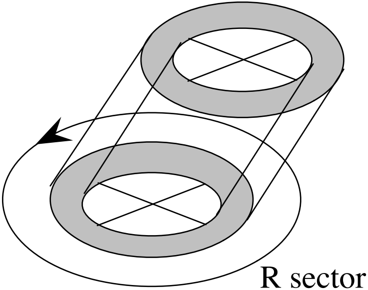
a solid torus with an embedded crosscap.

Let's apply it to $\mathbf{U(1)_2 \times U(1)_{-1}}$.

What is the state $|\text{crosscap}\rangle$ given by the following ?

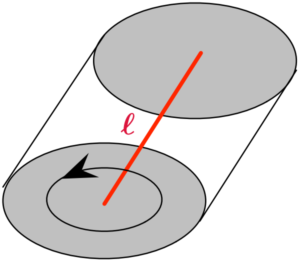


The horizontal direction is automatically in the R sector



since we go round the central crosscap twice.

So the state $|\text{crosscap}\rangle$ is a linear combination of $|\ell\rangle$



where ℓ is a line operator in the R-sector of $\mathbf{U(1)_2 \times U(1)_{-1}}$.

The line operators of $\mathbf{U(1)_2}$ are either

- trivial with $h = 0$
- nontrivial with $h = +1/4$

and the R-line operator of $\mathbf{U(1)_{-1}}$ has

- $h = -1/8$.

Combining them, there are only two states in the R-sector of $\mathbf{U(1)_2 \times U(1)_{-1}}$:

$$T|\ell\rangle = e^{+2\pi i 2/16}|\ell\rangle, \qquad T|\ell'\rangle = e^{-2\pi i 2/16}|\ell'\rangle.$$

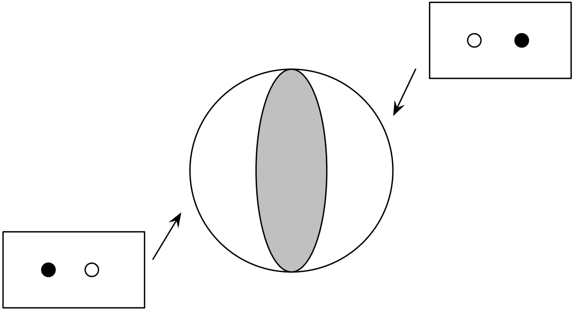
The state $|\text{crosscap}\rangle$ is a linear combination of them, and is a T eigenstate. So we have

$$|\text{crosscap}\rangle \propto |\ell\rangle \qquad \nu = +2$$

or

$$|\text{crosscap}\rangle \propto |\ell'\rangle \qquad \nu = -2.$$

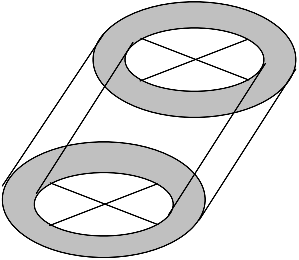
This value $\boldsymbol{\nu = \pm 2}$ is consistent with what we deduced from the domain wall construction:



where the bulk had $\boldsymbol{\nu{=}3}$ fermions and the domain wall had

a $\boldsymbol{\nu{=}1}$ goldstino $+$ $\mathbf{U(1)_2}$ CS.

For more complicated 2+1d TQFTs, the determination of $|\text{crosscap}\rangle$



is not this simple, but this can be done in many cases.

More details can be found in [YT-Yonekura, to appear soonish].

We know that an oriented 2+1d TQFT is specified by the data satisfying the **Moore-Seiberg axiom**.

Clearly, we need to have an
unoriented, spin version of the Moore-Seiberg axiom.

Then $\mathbb{Z}_{\mathbf{16}}$ classification of the time-reversal anomaly
would be an automatic outcome.

I would hope to work this out in the future.

Happy 60th anniversay, Nati!