On the time-reversal anomaly of 2+1d TQFTs

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in collaboration with
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It is a great honor to speak at Natifest.

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I got a high fever right after that, and was in bed for a few days.

That was 11 years ago. Time flies!
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For example, one cold winter day,
I overheard Davide Gaiotto chatting with Nati.

As they sounded very excited, I asked them if I could join.
It turned out that Davide was explaining to Nati what became known as the class S theory!

It was a few months before the first paper came out, and it gave me a head start working in this business.
Another episode:

In 2008, I gave a local seminar here, on a counterexample to the a-theorem I thought I found with Al Shapere. Nati didn’t like it.
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In 2008, I gave a local seminar here, on a counterexample to the a-theorem I thought I found with Al Shapere. Nati didn’t like it. At all.

Two years later, he told me he debugged it with Davide.

In the end it became a paper by Nati, Davide and me.

Soon the a-theorem was proved by Zohar and Adam Schwimmer.
So far I have three papers with Nati, and I learned a lot by working with him.

Of course I learned a lot about physics, but somehow I feel I learned more about the attitude toward physics.

For example:

- the importance of **finding the right question to ask**,  
- of **identifying the crucial elements in the answer**,  
- and **how to concisely express those elements in a paper**.

When he edits the draft, it often becomes **shorter** and **clearer**.
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that when I heared a rumor last summer that he and Edward were working on topological phases,
I remain such a loyal follower of Nati, that when I heared a rumor last summer that he and Edward were working on topological phases, I decided I should work on it too!

Today I’d like to say something about it. All is based on my collaboration with Kazuya Yonekura, a postdoc at IPMU and a former postdoc here at IAS.
Today I’d like to discuss

Anomaly of time-reversal symmetry of 2+1d systems

• What is it?
• What are some systems that have it?
• How should one determine it?
I’d like to first remind ourselves of a completely understood case of:

**Anomaly of \( U(1) \) symmetry of 3+1d systems**

- What is it?
- What are some systems that have it?
- How should one determine it?
Anomaly of $\mathbf{U}(1)$ symmetry of 3+1d systems:

What is it?

Phase ambiguity of the partition function in the presence of background $\mathbf{U}(1)$ gauge field.

The phase ambiguity occurs in a controlled way, as follows.
Consider 5d closed manifold $X$ with a background $U(1)$ field, with the CS action

$$\exp \left[ 2\pi i k \int_X A \wedge F \wedge F \right]$$

This is invariant under the gauge transformation if $k \in \mathbb{Z}$.

(I would be sloppy about the normalizations. Forgive me.)
If the 5d manifold $X$ has a boundary, $M = \partial X \neq \emptyset$, the CS action

$$\exp \left[2\pi i k \int_X A \wedge F \wedge F\right]$$

is not invariant even when $k \in \mathbb{Z}$, due to the gauge variation at the boundary.
You can add **something physical** at the boundary $M = \partial X$

so that the combination

$$Z_M[A|_M] \exp\left[2\pi ik \int_X A \wedge F \wedge F\right]$$

is **invariant**: the phase ambiguities of two terms cancel each other.

This is also called the anomaly inflow. [Callan-Harvey, …]
A typical example of such something physical is, of course, charged chiral fermions on $M$.

Is there another way to see such chiral fermions arise on the boundary?
Given $k$ charged massive 5d fermion with the mass term

$$m\psi \bar{\psi},$$

integrating them out generates the CS term

$$\exp[\pm 2\pi i \frac{k}{2} \int A \wedge F \wedge F]$$

where the sign $\pm$ is the sign of $m$. 
Now, instead of

\[ \exp \left[ 2\pi i k \int_X A \wedge F \wedge F \right] \]
We can consider

\[ \exp[+2\pi i \frac{k}{2} \int A \wedge F \wedge F] \]

\[ \exp[-2\pi i \frac{k}{2} \int A \wedge F \wedge F] \]
which you can represent as

\begin{center}
\begin{tikzpicture}
\node at (0,0) {\textit{k 5d fermions with mass }$m > 0$};
\node at (0,-3) {\textit{k 5d fermions with mass }$m < 0$};
\end{tikzpicture}
\end{center}
but when the fermion mass is space dependent

\[ m > 0 \]

\[ m < 0 \]
we know that there is a zero mode
so we have

\[ k \text{ 5d fermions with mass } m > 0 \]

\[ k \text{ 5d fermions with mass } m < 0 \]

\[ k \text{ 4d massless chiral fermions} \]
which is

\[
\exp \left[ 2\pi i k \int_X A \wedge F \wedge F \right]
\]

\(k\) 4d massless chiral fermions
Note the equality of pictures

\[ \exp \left[ 2\pi ik \int_X A \wedge F \wedge F \right] \]

\( k \) 4d massless chiral fermions

and

\( k \) 5d fermions with mass \( m > 0 \)

\( k \) 5d fermions with mass \( m < 0 \)

\( k \) 4d massless chiral fermions
So far we recalled

**Anomaly of $\textbf{U}(1)$ symmetry of 3+1d systems.**

But today I wanted to discuss

**Anomaly of time-reversal symmetry of 2+1d systems.**
A typical time-reversal invariant system in $3+1d$ is a massive Majorana fermion with the mass term

$$m\psi\psi$$

where $m \in \mathbb{R}$ to be invariant under the time reversal.
If we make $m$ space-dependent, we have a zero-mode

$$
\psi_m > 0 \quad m < 0
$$

which is a massless Majorana fermion in $2+1d$. 

$m > 0$

$m < 0$
So we can have the situation

\[ \nu \text{ 4d fermions with mass } m > 0 \]
\[ \nu \text{ 4d fermions with mass } m < 0 \]
\[ \nu \text{ 3d massless Majorana fermions} \]

where we consider general non-orientable manifolds, to give an equivalent of ‘background gauge field for time-reversal.’
We can integrate out the massive fermions to find

\[ \exp \left[ 2\pi i \nu \eta \right] \]

\( \nu \) 3d massless Majorana fermions

where \( \eta \) is the so-called eta invariant, that plays the role of the CS term in this case.

Condensed-matter theorists know the bulk by the name 3+1d topological superconductor.
On a closed (in general non-orientable) 4d manifold $X$, 

$$\exp [2\pi i \eta]$$

is always a 16th root of unity. Therefore, in the expression 

$$\exp [2\pi i \nu \eta]$$

only $\nu \in \mathbb{Z}_{16}$ matters.

This is known under the name \textbf{$\mathbb{Z}_{16}$-classification of the 3+1d topological superconductor.}
For us, this means that

**the time-reversal anomaly of 2+1d systems is a quantity** \( \nu \in \mathbb{Z}_{16} \).

For example, \( \nu \) massless 2+1d Majorana fermions have the value

\[
\nu \in \mathbb{Z}_{16}.
\]

What are other 2+1d systems with time-reversal anomalies?
Let’s start from

\[ \nu = 3 \] 4d fermions with mass \( m > 0 \)

\[ \nu = 3 \] 4d fermions with mass \( m < 0 \)

Regard 3 fermions as an adjoint of SU(2), and couple dynamical SU(2) gauge field to it.

This is \( \mathcal{N} = 1 \) SU(2) SYM softly broken by the gaugino mass.
So we have a domain wall between two vacua of $\mathcal{N}=1$ SU(2) SYM

on which it’s known that we have

\[ a \text{ 3d goldstino } \pm \text{ U}(1)_2 \text{ CS.} \]

[Acharya-Vafa,...]
We started from $\nu = 3$ 4d fermions, so

$$\nu[a \text{ 3d goldstino}] + \nu[U(1)_2 \text{ CS}] = 3.$$ 

Clearly

$$\nu[a \text{ 3d goldstino}] = 1.$$ 

Therefore

$$\nu[U(1)_2 \text{ CS}] = 2.$$ 

It is interesting that a theory without any massless things can have an anomaly. This can’t happen for the anomaly of continuous symmetry.
So, suppose we’re given a time-reversal symmetric 2+1d TQFT.

How do we determine its anomaly $\nu \in \mathbb{Z}_{16}$ directly?

[YT-Kazuya Yonekura, to appear soonish]
Consider putting the 3d theory on a crosscap:
Consider putting the 3d theory on a crosscap:

There is an isometry rotating it.
The associated conserved quantity is the momentum $p$. 
In a non-anomalous theory, we have

\[ p = n \in \mathbb{Z}. \]

This is because the \( 2\pi \) rotation should not do anything:

\[ \exp[2\pi i p] = 1. \]
In an anomalous theory, this might not hold, because of phase ambiguity:

$$\exp[2\pi ip] \neq 1.$$ 

This phase ambiguity should be ‘linear’ in $\nu$, so we have

$$\exp[2\pi ip] = \exp[2\pi ic\nu]$$

for some number $c$.

Equivalently,

$$p = n + c\nu, \quad n \in \mathbb{Z}.$$
We can fix \( c \) by considering a system whose \( \nu \) is known, for example a 3d Majorana fermion for which \( \nu = 1 \).

One finds by an explicit computation that \( c = 1/16 \).

[Hsieh-Cho-Ryu, 1503.01411]

Therefore:

\[
p = n + \frac{\nu}{16}, \quad n \in \mathbb{Z}.
\]

The time-reversal anomaly manifests itself as the anomalous momentum on the crosscap.

For 1+1d systems the same thing was pointed out in [Cho-Hsieh-Morimoto-Ryu, 1501.07285]
How do we compute the anomalous momentum for a 2+1d TQFT?

Suppose we have

\[ p = \frac{\nu}{16} \mod \mathbb{Z} \]

on
We have

\[ T|\text{crosscap}\rangle = e^{2\pi i\nu/16}|\text{crosscap}\rangle \]

where \( T \in \text{SL}(2, \mathbb{Z}) \) and \(|\text{crosscap}\rangle\) is a state on \( T^2 \) generated by a solid torus with an embedded crosscap.
Let’s apply it to $\mathbf{U}(1)_2 \times \mathbf{U}(1)_{-1}$.

What is the state $|\text{crosscap}\rangle$ given by the following?
The horizontal direction is automatically in the R sector since we go round the central crosscap twice.
So the state $|\text{crosscap}\rangle$ is a linear combination of $|\ell\rangle$

where $\ell$ is a line operator in the R-sector of $\mathbf{U}(1)_2 \times \mathbf{U}(1)_{-1}$. 

R sector

where $\ell$ is a line operator in the R-sector of $\mathbf{U}(1)_2 \times \mathbf{U}(1)_{-1}$.
The line operators of $\mathbf{U}(1)_2$ are either

- trivial with $h = 0$
- nontrivial with $h = +1/4$

and the R-line operator of $\mathbf{U}(1)_{-1}$ has

- $h = -1/8$. 
Combining them, there are only two states in the R-sector of 
\( \text{U}(1)_2 \times \text{U}(1)_{-1} \):

\[
T|\ell\rangle = e^{+2\pi i 2/16} |\ell\rangle, \quad T|\ell'\rangle = e^{-2\pi i 2/16} |\ell'\rangle.
\]

The state \(|\text{crosscap}\rangle\) is a linear combination of them, 
and is a \(T\) eigenstate. So we have

\[
|\text{crosscap}\rangle \propto |\ell\rangle \quad \nu = +2
\]

or

\[
|\text{crosscap}\rangle \propto |\ell'\rangle \quad \nu = -2.
\]
This value $\nu = \pm 2$ is consistent with what we deduced from the domain wall construction:

where the bulk had $\nu=3$ fermions and the domain wall had

\[ a \nu=1 \text{ goldstino} + U(1)_2 \text{ CS}. \]
For more complicated 2+1d TQFTs, the determination of $|\text{crosscap}\rangle$ is not this simple, but this can be done in many cases.

More details can be found in [YT-Yonekura, to appear soonish].
We know that an oriented 2+1d TQFT is specified by the data satisfying the Moore-Seiberg axiom.

Clearly, we need to have an unoriented, spin version of the Moore-Seiberg axiom.

Then $\mathbb{Z}_{16}$ classification of the time-reversal anomaly would be an automatic outcome.

I would hope to work this out in the future.
Happy 60th anniversary, Nati!