Comments on Seiberg Duality

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Natifest
September 16, 2016

Introduction

In this lecture I'll discuss some aspects of d=4, N=1 SUSY dynamics. Of course, one of Nati's many achievements is the elucidation of the low energy dynamics of N=1 SQCD, from his work with Affleck and Dine in the 1980's to the full picture that appeared in 1994.

However, this is not a historical talk. As I'll discuss, Nati's results, and some subsequent ones inspired by his work, lead to new questions that remain open today, and there are reasons to believe that there are qualitatively new phenomena waiting to be uncovered, and new techniques that are needed to uncover them.

To set the stage, I will start with a brief review of the SQCD case. We have a gauge theory with gauge group $SU(N_c)$ and N_f flavors of chiral superfields in the fundamental representation. Nati and collaborators showed that the low energy dynamics of this theory varies with N_f, N_c as follows:

The discrete parameter $x = \frac{N_c}{N_f}$ can be thought of as a `t Hooft coupling that measures the strength of gauge interactions in the IR.

- For $x \leq \frac{1}{3}$ the theory is IR free, like QED with massless electrons.

- For $x > \frac{1}{3}$ the gauge interactions are non-vanishing in the IR, and the theory approaches a non-trivial fixed point. For $x \simeq \frac{1}{3}$ the IR theory is weakly coupled (Banks-Zaks). In particular, the superconformal $U(1)_R$ current of the IR theory must be conserved throughout the RG flow. Thus, it can be identified in the UV, and one can use it to compute IR scaling dimensions of chiral operators as well as the central charges a, c.

- As the coupling x increases, the RG flow becomes ``longer'' and new phenomena are possible. For example, additional currents can come down to dimension 3 and the superconformal $U(1)_R$ can now be a linear combination of the ones visible in the UV and the new ones.

- One example of this is the decoupling of gauge invariant chiral operators whose IR dimension naively dips below one (the unitarity bound). This is easy to take into account, by allowing the UV $U(1)_R$ to mix with the currents associated with these free fields.

- It is not hard to see that this can't be it. For example, if nothing else happens, some RG flows in SQCD can be shown to violate the a-theorem at sufficiently large x . Thus, something new must happen at strong coupling to restore unitarity.

In SQCD that something is Seiberg duality.

This is the assertion that the infrared limit of the theory described above is the same as that of another theory, with gauge group $SU(N_f - N_c)$, similar charged matter, and singlet meson fields M, dual to the electric gauge invariant chiral operators $\tilde{Q}Q$, which are coupled to the magnetic quarks $\tilde{q}, q$ via the superpotential

$$W = M\tilde{q}q$$

The rank of the magnetic gauge group implies that the magnetic `t Hooft coupling is

$$x_m = \frac{N_f - N_c}{N_f} = 1 - x$$

Thus, as the electric theory becomes more strongly coupled, the magnetic one becomes more weakly coupled. The conformal window, $\frac{1}{3} \leq x \leq \frac{2}{3}$, is a region of overlap of the two descriptions. Outside of it, one or the other breaks down. That’s the origin of the apparent violations of the a-theorem.

Generalizations

- It is natural to ask whether any qualitatively new phenomena occur when one extends the discussion to other theories. One can consider other gauge groups, more general matter content, turn on superpotentials, etc. E.g. in string theory contexts it is natural to consider quiver theories, and there has been a lot of work on that.

- We here will keep the gauge group the same, $SU(N_c)$ (or $U(N_c)$), and consider what happens when we add matter in other representations.

Consider adding to the theory a chiral superfield X which transforms in the adjoint representation of the gauge group. There are a number of cases to consider:

- W=0 (no superpotential): in this case the infrared $U(1)_R$ current in the Banks-Zaks regime $\left(x \simeq \frac{1}{2}\right)$ is not uniquely determined. It can be determined by a-maximization (Intriligator-Wecht; DK, Parnachev, Sahakian; 2003). This theory is believed to flow to a non-trivial IR fixed point for all $x \geq \frac{1}{2}$. The infrared $U(1)_R$ current is essentially the one read off from that analysis.

- $W = Tr\, X^3$: this case is analogous to the SQCD one. For $x \simeq \frac{1}{2}$ the theory is weakly coupled in the IR and a Bankz-Zaks type analysis is valid. For $x > 2$ the theory exhibits runaway behavior. The strong coupling region is described in terms of a dual theory with gauge group $G = SU(2N_f - N_c)$, similar charged matter $q, \tilde{q}, \hat{X}$, and a magnetic superpotential

$$W = Tr\, \hat{X}^3 + M_1 \tilde{q} \hat{X} q + M_2 \tilde{q} q$$

DK

The singlet mesons M_1, M_2 correspond under the duality to the electric meson fields $\tilde{Q}Q$, $\tilde{Q}XQ$, respectively. Note that it is important that the superpotential gives rise to the F-term constraint $X^2 = 0$, which truncates the chiral ring.

- $W = Tr\, X^{k+1}$: for $k > 2$ the superpotential is naively irrelevant, but one can show that for large enough x gauge interactions make it relevant and it leads to a non-trivial IR fixed point. This fixed point exists for $x \leq k$, and towards the top of that range is better described by a dual theory, with gauge group $SU(kN_f - N_c)$, similar charged matter and k singlet meson fields M_j, that correspond to the electric mesons $\tilde{Q}X^{j-1}Q$, with $j = 1, \dots, k$.

  DK, A. Schwimmer, N. Seiberg

➢ An important aspect of this duality is the F-term equation $X^k = 0$, which truncates the chiral ring.

➢ The duality is again a strong-weak coupling one, as can be seen from the form of the magnetic `t Hooft coupling:

$$x_m = k - x$$

To summarize, the infrared behavior of theories with unitary gauge groups, any number of fundamentals and a single adjoint chiral superfield is understood, using a combination of supersymmetric Banks-Zaks (or, equivalently, NSVZ), a-maximization and Seiberg duality.

What happens when we add more adjoints? Since for 3 or more adjoints one in general finds free theories in the IR, the one remaining case is 2 adjoints. As we will see next, this case presents significant new challenges.

Two adjoints

We have gauge group $SU(N_c)$, N_f fundamentals $Q, \tilde{Q}$, and two adjoints X, Y. We can try to repeat the discussion of the previous case for this one, following Intriligator and Wecht (2003).

- $W = 0$: the theory approaches a non-trivial IR fixed point for all $x > 1$. This fixed point can be analyzed by using a-maximization.

- $W = Tr\, Y^3$: same as previous case.

If we want to stick to renormalizable superpotentials, the only remaining case is a cubic superpotential

$$W = Tr(X^3 + Y^3)$$

As we will see next, this case is quite puzzling.

The F-term equations are in this case

$$X^2 = 0; \;\; Y^2 = 0$$

Thus, chiral operators take the form

$$\tilde{Q}XYXY \ldots XQ$$
$$\tilde{Q}XYXY \ldots YQ$$
$$\tilde{Q}YXYX \ldots XQ$$
$$\tilde{Q}YXYX \ldots YQ$$

$$Tr\,(XY)^n \;\; ; \; n = 1,2, \ldots$$

etc. Note in particular that the ring is not truncated as in the single adjoint case above.

To study the IR behavior of the theory we need to determine the superconformal $U(1)_R$. For weak coupling, i.e. $x \simeq 1$, it must be the current conserved throughout the RG flow, that assigns charges

$$R_X = R_Y = \frac{2}{3}$$

$$R_Q = 1 - \frac{x}{3}$$

to the different fields. As x increases, there is the possibility of mixing of this current with other ones. Can it be that the only new effect that needs to be taken into account is decoupling of free fields, as in the previous two cases?

No – it is inconsistent with the a-theorem.

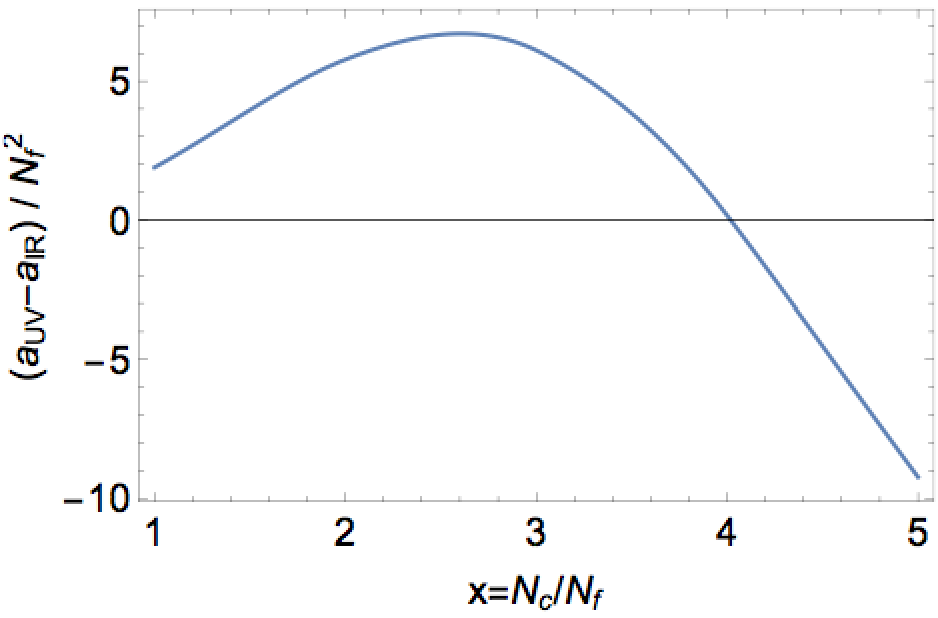
There are a number of ways of seeing that. One is to study the dependence of the central charge a on N_f for fixed N_c. One finds that the behavior is not monotonic (for large enough x), as one would expect.

Another way is to consider the flat direction corresponding to the solution of the F and D-term equations

$$X = b\sigma^{+} \otimes I$$
$$Y = b^{*}\sigma^{-} \otimes I$$

K. Intriligator, E. Nardoni

This flat direction describes an RG flow from the original theory to one with gauge group $SU\left(\frac{N_c}{2}\right)$, $2N_f$ flavors and a single adjoint field with vanishing superpotential. The change of the central charge a along this flow is given by



Thus, something else must be going on.

In all the previous examples, the region in which the original description broke down was described by a Seiberg dual theory. However, in those examples the chiral ring was always truncated. Theories in which the chiral ring was not truncated did not suffer from violations of the a-theorem and did not require Seiberg duality.

The two adjoint theory with $W = Tr(X^3 + Y^3)$ has some elements of each of the two classes of theories. On the one hand it seems to require something like Seiberg duality to avoid violations of unitarity; on the other, the chiral ring is not truncated and thus is sensitive to the rank of the gauge group, which makes such a duality difficult.

Brodie duality

As it happens, a generalization of Seiberg duality to this case was proposed in 1996 by J. Brodie. He discussed a more general class of theories, corresponding to superpotentials of the form

$$W = Tr\left(\frac{1}{k+1}X^{k+1} + XY^2\right)$$

For k=2, this superpotential is equivalent to the one we wrote before by the transformation $X \to X + Y; Y \to X - Y$.

The key point of Brodie's work is that for odd k the F-term equations of this superpotential imply the constraint $Y^3 = 0$. This truncates the chiral ring to a set of $3k$ mesons

$$M_{lj} = \tilde{Q}X^{l-1}Y^{j-1}Q;\;\; l = 1,\dots,k;\;\; j = 1,2,3$$

which appear in the dual theory as singlet fields coupled to the analogous magnetic meson fields.

Correspondingly, the dual gauge group is $SU(3kN_f - N_c)$.

For even k, such as $k = 2$, the case discussed above, the constraint $Y^3 = 0$ is absent, at least classically. Thus, the chiral ring is not truncated, and the duality seems suspect. At the same time, it seems that something like this duality is necessary to resolve the tensions mentioned above.

What's going on?

To try to make progress, consider the deformation

$$W = Tr\left(\frac{1}{k+1}X^{k+1} + XY^2\right) + \lambda\tilde{Q}Y^3Q$$

The term proportional to λ is interesting because it imposes (via the F-terms of Q, $\tilde{Q}$) the constraints necessary for Brodie duality, $Y^3Q = 0$, etc. Thus, in the presence of this term it makes sense to assume this duality, and we will do that below.

The R-charge of λ is

$$R(\lambda) = \frac{2x - 3k}{k+1}$$

In particular, it is marginal at $x = \frac{3k}{2}$, the self-dual point under Brodie duality. There is a conformal manifold associated with it.

To understand the physics associated with this coupling, it is useful to discuss the analogous coupling in the cases that are understood, namely A_k and D_{k+2} with odd k.

- A_k: $$W = Tr\, X^{k+1}$$

F-term condition for X: $X^k = 0$

Deformation: $W = \frac{1}{k+1} Tr\, X^{k+1} + \lambda\, \tilde{Q} X^k Q$

One might think that the λ deformation is trivial, since the operator $\tilde{Q} X^k Q$ is non-chiral, but in fact it isn't.

A quick way to see that is to change variables:

$$X \to X - \lambda Q\tilde{Q}$$

(where $Q\tilde{Q}$ is an adjoint of $U(N_c)$ and a singlet of $U(N_f)$)

This eliminates the linear term in λ but generates higher order terms. One gets

$$W = \frac{1}{k+1} Tr\, X^{k+1} - \frac{\lambda^2}{2} \sum_{j=1}^{k} M_j\, M_{k+1-j} + O(M^3)$$

where

$$M_{\mathrm{j}} = \tilde{Q} X^{j-1} Q$$

The R-charge of λ is

$$R(\lambda) = \frac{4x - 2k}{k + 1}$$

It is marginal at the self-dual point under the A_k duality $x \to k - x$. For $x < \frac{k}{2}$ the λ perturbation is irrelevant, and turns off in the infrared. For $x > \frac{k}{2}$ it is relevant and grows.

In the dual magnetic variables, the λ perturbation is quadratic in the singlet meson fields. Hence one can integrate them out, and obtain an effective quartic superpotential in terms of the magnetic quarks,

$$W = \frac{1}{k+1} Tr\, \hat{X}^{k+1} - \frac{1}{2\lambda^2} \sum_{j=1}^{k} \widehat{M}_j \widehat{M}_{k+1-j} + O(\widehat{M}^3)$$

with

$$\widehat{M}_{\mathrm{j}} = \tilde{q} \hat{X}^{j-1} q$$

Thus, we see that for $x > \frac{k}{2}$, where in the electric variables the λ deformation grows in the IR, in the magnetic ones it goes to zero. So, the infrared limit of the electric theory is the magnetic one, with no singlet mesons and vanishing superpotential.

For $x = \frac{k}{2}$ we have a conformal manifold, and duality implies a relation between small and large λ.

- D_k: Something similar happens for the D_k case with odd k. Shifting variables:

$$X \to X - \frac{\lambda}{2}\left(\mathrm{Y}Q\tilde{Q} + Q\tilde{Q}Y\right)$$

$$Y \to Y + \frac{\lambda}{2}\left(X^{k-1}Q\tilde{Q} - X^{k-2}Q\tilde{Q}X + \cdots + Q\tilde{Q}X^{k-1}\right)$$

One finds that the λ deformation is equivalent to

$$\delta W = -\left(\frac{\lambda}{2}\right)^2 \sum_l \left(2(-)^{l+1} M_{l,1} M_{k+1-l,3} + M_{l,2} M_{k+1-l,2}\right) + \cdots$$

For x below the self dual point this perturbation turns off in the infrared, while above it the theory flows to the magnetic theory.

We are now ready to get back to the problematic case, D_k with k even. It is natural to conjecture that in this case too the infrared limit of the theory with the λ deformation for x slightly above $\frac{3k}{2}$ (and at large λ for $x = \frac{3k}{2}$) is the magnetic Brodie theory with the singlet mesons integrated out.

But now, in the electric variables the chiral ring satisfies the constraints postulated by Brodie, and therefore the magnetic one must too.

These considerations lead one to the following picture: the theories with

$$W = Tr\left(\frac{1}{k+1}X^{k+1} + XY^2\right)$$

have the property that in a region around $x = \frac{3k}{2}$ they satisfy the quantum constraint

$$Y^3 = 0$$

Outside of this region, this constrained is absent. This scenario resolves the problems I mentioned with the even k theories, and might help with some other ones that I did not mention.

Outside of this region, this constraint is absent. This scenario resolves the problems I mentioned with the even k theories, and might help with some other ones that I did not mention.

Of course, if true an important open problem is to calculate the boundaries of this region. It is not known at present how to do that.

Conclusions

- The infrared behavior of N=1 SUSY theories in 4d is a problem that still has a lot to teach us about dynamical mechanisms in QFT.

- We need more insights from Nati to make progress on it.

HAPPY BIRTHDAY, NATI!