SOME MODULAR PROPERTIES OF SUPERSTRING SCATTERING AMPLITUDES

Michael B. Green, University of Cambridge
Nati –

Thank you for the elegance of your insights conveyed with such enthusiasm.

**keep enthusising for a long time to come.**
GENERAL SETTING

TO WHAT EXTENT DO DUALITY AND SUPERSYMMETRY CONSTRAIN THEORIES WITH A LARGE AMOUNT OF SUPERSYMMETRY? e.g. Maximal supergravity/Type II string theory

THE LOW ENERGY EXPANSION OF STRING AMPLITUDES

Consider narrowly-focused aspects of the low energy expansion of closed string theory obtained from maximally supersymmetric closed string scattering amplitudes.

- **EXPlicit Features of Low Order Type II String Perturbation Theory**
  - Modular invariants of Riemann surfaces
  - Mathematical connections to **MULTIPLE-ZETA VALUES** and their **ELLIPTIC GENERALISATIONS**

    With: Eric D’Hoker; Pierre Vanhove; Omer Gurdogan

    Recent papers
    - 1502.06698
    - 1509.00363
    - 1512.06779
    - 1603.00839

- **Part of a Larger Programme Investigating**

- **Non-Perturbative Features of String Amplitudes**
  - Constraints imposed by SUSY, Duality, Unitarity
  - Connects perturbative with non-perturbative effects
    - Modular Forms; Automorphic forms for higher-rank groups; …. 
  - Coefficients of BPS interactions encoding BPS microstate-counting

earlier work:
- Stephen Miller; Don Zagier;
- Boris Pioline; Jorge Russo;
- Rudolfo Russo; Carlos Mafra;
- Oliver Schlotterer; Anirban Basu;
- Sav Sethi, Michael Gutperle, …..
**Four-Graviton Scattering in Type IIB String Theory**

\[ A^{(4)}(\epsilon_r, k_r; \Omega) = R^4 \, T^{(4)}(s, t, u; \Omega) \]

\[ \mathcal{R} \text{ linearized curvature } \sim k_\mu k_\nu \epsilon_{\rho\sigma} \]

One complex modulus

\[ \Omega = \Omega_1 + i\Omega_2 \]

\[ \Omega_2 = \frac{1}{g} = e^{-\phi} \]

inverse string coupling constant

Symmetric function of Mandelstam invariants \( s, t, u \) (with \( s + t + u = 0 \)).

Has an expansion in power series of \( \sigma_2 = s^2 + t^2 + u^2 \) and \( \sigma_3 = s^3 + t^3 + u^3 \)

*(Non-analytic pieces are essential, but will be ignored in this talk)*

\[ T(s, t, u; \Omega) = \sum_{p, q} \mathcal{E}_{(p, q)}(\Omega) \, \sigma_2^p \sigma_3^q \]

\[ \sim s^{2p+3q} + \ldots \]

Coefficients are \( SL(2, \mathbb{Z}) \)-invariant functions of scalar fields (moduli, or coupling constants).

**To what extent can we determine these coefficients?**

**Boundary data:** String Perturbation Theory

\[ \Omega_2 \to \infty \quad (g \to 0) \]
**TREE-LEVEL** ("**VIRASORO**" AMPLITUDE)

\[ A_0^{(4)}(\epsilon_r, k_r) = g^{-2} \mathcal{R}^4 T_0^{(4)}(s, t, u) \]

\[ T_0^{(4)} = \frac{1}{stu} \frac{\Gamma(1 - \alpha's) \Gamma(1 - \alpha't) \Gamma(1 - \alpha'u)}{\Gamma(1 + \alpha's) \Gamma(1 + \alpha't) \Gamma(1 + \alpha'u)} = \frac{3}{\sigma_3} \exp \left[ \sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{2n+1} \alpha'^{2n+1} \sigma_{2n+1} \right] \]

\[ \sigma_n = s^n + t^n + u^n \]

**Tree-level SUPERGRAVITY**

\[ s^k \mathcal{R}^4 \sim d^{2k} \mathcal{R}^4 \]

\[
\begin{align*}
T_0^{(4)} &= \frac{3}{\sigma_3} + 2\zeta(3) \alpha^3 + \zeta(5) \alpha^5 \sigma_2 + \frac{2\zeta(3)^2}{3} \alpha'^6 \sigma_3 + \frac{\zeta(7)}{2} \alpha'^7 \sigma_2^2 \\
&\quad+ \frac{2\zeta(3)\zeta(5)}{3} \alpha'^8 \sigma_2 \sigma_3 + \frac{\zeta(9)}{4} \alpha'^8 \sigma_2^3 + \frac{2}{27} (2\zeta(3)^2 + \zeta(9)) \alpha'^9 \sigma_3^2 + \ldots
\end{align*}
\]

\[ \sigma_2 = s^2 + t^2 + u^2 \]

\[ \sigma_3 = s^3 + t^3 + u^3 \]

**INFINITE SERIES** of \( d^{2k} \mathcal{R}^4 \) terms. **COEFFICIENTS ARE POWERS OF ODD RIEMANN \( \zeta \) VALUES** with rational coefficients.

Generalisation to N-particle scattering involves multiple zeta values.
**ZETA VALUES AND MULTIPLE-ZETA VALUES**

**ZETA VALUES:**
- Special values of POLYLOGARITHMS
  \[ Li_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a} \]
  \[ \zeta(a) = Li_a(1) \]

  Even zeta values \( \zeta(2n) = c_n \pi^{2n} \)
  Odd zeta values \( \zeta(2n+1) \) transcendental?

**MULTI-ZETA VALUES (MZV’s):**
- Special values of MULTIPLE POLYLOGARITHMS
  \[ Li_{a_1,\ldots,a_r}(z_1,\ldots,z_r) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} \left( \frac{z_{\ell}}{k_{\ell}} \right)^{a_{\ell}} \]
  \[ \zeta(a_1,\ldots,a_r) = Li_{a_1,\ldots,a_r}(1,\ldots,1) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} k_{\ell}^{-a_{\ell}} \]
  “weight” \( w = \sum_{\ell=1}^{r} a_{\ell} \)  “depth” \( r \)

  MZV are numbers with algebraic properties inherited from the algebraic properties of multiple polylogarithms – “STUFFLE” and “SHUFFLE” relations.
  e.g. first non-trivial (irreducible) case is weight \( w = 8 \)

  \[ 350 \zeta(3, 5) = 875 \zeta(6, 2) + 240 \zeta(2)^4 - 1400 \zeta(3) \zeta(5) \]

  • **THE DIMENSION** \( d_w \) **OF THE SUBSPACE OF MZV’s OF WEIGHT** \( w \) **OVER** \( \mathbb{Q} \)
  \[ \sum_{w=0}^{\infty} d_w x^w = \frac{1}{1 - x^2 - x^3} \]
**N-PARTICLE TREE AMPLITUDES**

**OPEN-STRING TREES:** For $N > 4$ coefficients of higher derivative interactions of order $\alpha'^n$ are multiple zeta values with weight $n$ (Yang-Mills) (Stieberger, Broedel, Mafra, Schlotterer)

**CLOSED-STRING TREES:** For $N > 4$ coefficients are *single-valued* MZV’s (svMZV’s) (gravity) (Brown) (Schlotterer, Stieberger)

- Special values of *single-valued* multiple polylogarithms – NO MONODROMIES (generalisations of BLOCH-WIGNER dilogarithm $\text{Im} \left( \text{Li}_2(z) + \log(1 - z) \log |z| \right)$)

- Kills even zeta values $\zeta_{sv}(2n) = 0$ Also $\zeta_{sv}(2n + 1) = 2\zeta(2n + 1)$ - ODD ZETA’S ONLY

- First non-trivial case is $\zeta_{sv}(3, 5, 3) = 2\zeta(3, 5, 3) - 2\zeta(3)\zeta(3, 5) - 10\zeta(3)^2\zeta(5)$
  weight $w = 11$

- Role of the KLT construction?

**HOW DOES THIS GENERALIZE TO HIGHER GENUS ??**
**Genus One Amplitude**

\[ \mathcal{A}_1^{(4)}(\epsilon_r, k_r) = \frac{\pi}{16} \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d\tau^2}{y^2} B_1(s, t, u; \tau) \]

Integral over complex structure \( \tau = x + iy \)

\[ B_1(s, t, u; \tau) = \frac{1}{y^4} \int_{\Sigma^4} \prod_{i=1}^{4} d^2 z \exp \left( -\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j G(z_i, z_j) \right) \]

Vertex operator Corr. function

Green function

Low energy expansion - integrate powers of the genus-one Green function over the torus and over the modulus of the torus – difficult!

(MBG, D'Hoker, Russo, Vanhove)

Expanding in a power series in momenta gives (with \( \alpha' = 4 \))

\[ \frac{1}{w!} \frac{1}{y^4} \int_{\Sigma^4} \prod_{i=1}^{4} d^2 z_i \left( \sum_{0<i<j\leq 4} s_{ij} G(z_i - z_j) \right)^w = \sum_i \sigma_2^{p_i} \sigma_3^{q_i} j^{(p_i, q_i)}(\tau) \]

Coefficients of higher derivative interactions

\[ \Xi(p, q) = \int_{\mathcal{M}_1} \frac{d^2 \tau}{y^2} j^{(p, q)}(\tau) \]

Coefficients of higher derivative interactions:

(genus-one generalisation of the tree-level values)
"Modular Graph Functions"

\( j^{(p,q)}(\tau) \) is sum of world-sheet Feynman diagrams.

Each of these is a modular function - invariant under \( SL(2, \mathbb{Z}) \)

The Green function on a torus of complex structure

\[
G(z) = -\ln \left| \frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} \right|^2 - \frac{\pi}{2y} (z - \bar{z})^2
\]

\[ z = u + \tau v \]

doubly periodic function

\[
\sum_{(m,n) \neq (0,0)} \hat{G}(m, n) e^{2\pi i (mu - nv)} + 2 \ln \left( \frac{2\pi}{|\eta(\tau)|^2} \right)
\]

**Momentum-space Propagator:**

integer world-sheet momenta \( m, n \in \mathbb{Z} \)

\[
\hat{G}(m, n) = \frac{y}{|m\tau + n|^2}
\]

General contribution to 4-particle amplitude: \( i, j = 1, 2, 3, 4 \)

Modular function

\[
D_{\ell_1, \ell_2, \ell_3, \ell_4; \ell_5, \ell_6} = \]

\( \ell_5 \) labels number of propagators on line \( S \)

"Weight" \( w = \ell_1 + \ell_2 + \cdots + \ell_6 \) contributes to \( D^2 w \mathcal{R}^4 \)
Multiple sums:

\[ D^4 \mathcal{R}^4 = \sum_{(m,n) \neq (0,0)} \frac{y^2}{|m\tau + n|^4} \equiv E_2(\tau) \]

\[ E_s(\tau) = \sum_{(m,n) \neq (0,0)} \frac{y^s}{|m\tau + n|^{2s}} \]

**Non-holomorphic SL(2) Eisenstein series**

\[ C_{a,b,c}(\tau) = \sum_{(m_1, n_1) \neq (0,0)} \frac{y^{a+b+c}}{|m_1\tau + n_1|^{2a}|m_2\tau + n_2|^{2b}|m_3\tau + n_3|^{2c}} \]

**World-sheet Feynman diagrams**

\[ C_{1,1,1} \equiv D_3 \]
\[ C_{2,2,1} \equiv D_{1,1,1,1,1} \]
\[ C_{3,1,1} \equiv D_{2,1,1,1} \]
\[ C_{4,3,2} \equiv D_{18} \mathcal{R}^4 \]

\[ w = a + b + c \]
\[ (w - 1) \text{ vertices} \]

(two-loop diagrams)
Direct analysis looks forbidding. But these functions satisfy simple Laplace equations with Laplacian
\[ \Delta = y^2 \left( \partial_x^2 + \partial_y^2 \right) \]

Simple examples of LAPLACE EQUATIONS:

\[ w = 3 \]
\[ \Delta \left( C_{1,1,1} - E_3 \right) = 0 \]

**SOLUTION:**
\[ C_{1,1,1} = E_3 + \zeta(3) \quad \text{(also Zagier)} \]

\[ w = 4 \]
\[ (\Delta - 2) C_{2,1,1} = 9E_4 - E_2^2 \]

\[ w = 5 \]
\[ (\Delta - 6) C_{3,1,1} = \frac{6}{5}E_5 + \frac{\zeta(5)}{10} + 16E_5 - 4E_2 E_3 \]

\[ w > 5 \]
Degeneracy – simultaneous inhomogeneous Laplace eigenvalue equations.
COEFFICIENTS OF $D^8 \mathcal{R}^4$ (WEIGHT-4)

$D_2^2 = E_2^2$  \hspace{1cm} $D_4$  \hspace{1cm} $D_{2,1,1} \equiv C_{2,1,1}$  \hspace{1cm} $D_{1,1,1,1}$

COEFFICIENTS OF $D^{10} \mathcal{R}^4$ (WEIGHT-5)

$D_5$  \hspace{1cm} $D_{2,2,1}$  \hspace{1cm} $D_{3,1,1}$  \hspace{1cm} $D_{1,1,1,1;1} \equiv C_{2,2,1}$

$D_{2,1,1,1} \equiv C_{3,1,1}$  \hspace{1cm} $D_{1,1,1} D_2$  \hspace{1cm} $D_3 D_2$
**Relation to Single-Valued Elliptic Multiple Polylogarithms**

(D’Hoker, MBG, Gurdogan, Vanhove)

**A Modular Graph Function is a Single-Valued Elliptic Multiple Polylogarithm Evaluated at a Special Value of its Argument**

A typical Modular Graph Function:

$$D(\tau) = \int d^2 z_1 \ldots d^2 z_4$$

i.e. Split one vertex of a modular graph function and leave it UNINTEGRATED

Now Consider

$$\tilde{D}(\zeta; \tau) = \int d^2 z_2 d^2 z_3 d^2 z_4$$

with \( \zeta = \exp(2\pi i (z_5 - z_1)) \)

It is easy to see that

$$D(\tau) = \tilde{D}(1; \tau)$$

- \( \tilde{D}(\zeta; \tau) \) is single valued (in \( \zeta \)) elliptic multiple polylogarithm
- Generalisation of single-valued elliptic polylogarithm of Zagier (1990)

$$D_{a,b}(\zeta; \tau) = \frac{(2i\tau_2)^{a+b-1}}{2i\pi} \sum_{(m,n)\neq(0,0)} e^{2i\pi(nu-mv)} \frac{1}{(m\tau + n)^a(m\bar{\tau} + n)^b}$$
MODULAR GRAPH FUNCTIONS OF ARBITRARY WEIGHT

Special values of SINGLE-VALUED ELLIPTIC MULTIPLE POLYLOGARITHMS

As with MZV’s, these elliptic functions satisfy a fascinating set of polynomial relationships – we have found a few of these (with great difficulty!)

POLYNOMIAL RELATIONSHIPS

A general modular graph function has a q-expansion with a finite number of powers of $\tau_2$

- q-EXPANSIONS

$$D_{...}(q, \bar{q}) = \sum_{k=-w}^{w-1} \tau_2^{-k} F_k(q, \bar{q})$$

$$F_k(q, \bar{q}) = \sum_{m,n=0}^{\infty} c_{mn} q^m \bar{q}^n$$

- The LAURENT SERIES ($m = n = 0$) dominates in the large-$\tau_2$ limit.

$$D_{...}^{Laurent} = \sum_{k=-w}^{w-1} c_{00}^k \tau_2^{-k}$$

- The coefficients of the Laurent series $c_{00}^k$ are rational multiples of MULTIPLE ZETA VALUES. Determined, by analysing the $\tau_2 \to \infty$ asymptotics.
Specific polynomials (over $\mathbb{Q}$) of modular graph functions with a given weight have vanishing Laurent series (making use of MZV algebraic relations).

It turns out (in every example we have studied): **WHEN THE LAURENT SERIES OF THE POLYNOMIAL VANISHES THE EXACT POLYNOMIAL VANISHES.**

Is this always the case — is this a theorem?

\[
C_{1,1,5}(y) = \frac{34}{8513505} y^7 + \frac{2}{945} \zeta(3) y^4 + \frac{17}{252} \zeta(5) y^2 + \frac{23}{105} \zeta(3) y + \frac{1391}{560} \zeta(7)
\]

\[
- \frac{3 \zeta(3) \zeta(5)}{y} + \frac{953 \zeta(9) + 144 \zeta(3)^3}{32 y^3} - \frac{1701 \zeta(3) \zeta(7) + 120 \zeta(5)^2}{32 y^3}
\]

\[
+ \frac{324 \zeta(3, 5, 3) - 324 \zeta(3) \zeta(3, 5) + 22299 \zeta(11) + 8460 \zeta(5) \zeta(3)^2}{320 y^4}
\]

\[
- \frac{891 \zeta(5) \zeta(7) + 702 \zeta(9) \zeta(3)}{16 y^5} + \frac{7209 \zeta(13)}{128 y^6},
\]

(Zerbini)
EXAMPLES OF POLYNOMIAL RELATIONSHIPS

e.g. weight 5

\[ D_5 - 60 C_{3,1,1} - 10 E_2 C_{1,1,1} + 48 E_5 - 16 \zeta(5) = 0 \]

polynomial of weight 5 in functions of different depth (different no. of loops).

e.g. weight 6

\[ -3 D_{411} + 109 C_{222} + 408 C_{321} + 36 C_{411} + 18 C_{211} E_2 + 12 E_3^2 - 211 E_6 + 12 E_3 \zeta_3 = 0 \]

polynomial of weight 6 in functions of different depth.

PROOF USES THE FOLLOWING EXPERIMENTAL FACT:

Consider: \( \mathcal{F}_w = \text{sum of weight-} \ w \text{ modular graph functions with vanishing Laurent series} \) i.e. \( \lim_{\tau_2 \to \infty} \mathcal{F}_w \sim e^{-\tau_2} \)

THEN: \( \nabla^{w-1} \mathcal{F}_w = 0 \quad \Rightarrow \quad \mathcal{F}_w = 0 \) (See also alternative proof by Basu)

where \( \nabla = \tau_2^2 \frac{\partial}{\partial \tau} \)
**Conjecture:**

**Modular graph functions of a given weight satisfy polynomial relations with rational coefficients**

Elliptic generalisation of the rational polynomial relations between multiple polylogarithms and single-valued MZV’s

**Question:**

**What is the basis of modular graph functions?**

- Some (presumably) related issues in open string loop amplitudes (Broedel, Mafra, Matthes, Schlotterer), which involve "holomorphic" elliptic multiple polylogarithms (Brown, Levin).

- **Important generalisation to modular graph forms**
INTEGRATION OVER FUNDAMENTAL DOMAIN

GENUS-ONE EXPANSION COEFFICIENTS:

Integrating over $\tau$ - using the earlier relations - gives the one-loop expansion:

$$A_{1}^{(4)} = \frac{\pi}{3} \left( 1 + 0 \sigma_{2} + \frac{\zeta(3)}{3} \sigma_{3} + 0 \sigma_{2}^{2} + \frac{116 \zeta(5)}{5} \sigma_{2} \sigma_{3} \ldots \right) \mathcal{R}^{4}$$

These coefficients are analogous to the tree-level coefficients:

WHAT IS THE CONNECTION BETWEEN THEM?
**Genus Two**

Amplitude is explicit but difficult to study.

(D’Hoker, Gutperle, Phong)

Low energy expansion:

(D’Hoker, MBG, Pioline, R. Russo)

Result:

\[ A_2^{(4)} = g_s^2 \left( \frac{4}{3} \zeta(4) \sigma_2 R^4 + 4\zeta(4)\sigma_3 R^4 + \ldots \right) \]

\[ d^4 R^4 \quad d^6 R^4 \]

**Genus Three**

Technical difficulties analysing 3-loops. Gomez and Mafra evaluated the leading low energy behaviour using PURE SPINOR FORMALISM, giving

\[ A_3^{(4)} = g_s^4 \left( \frac{4}{27} \zeta(6) \sigma_3 + \ldots \right) R^4 \]

\[ d^6 R^4 \]

**Higher Orders**

New problems - No explicit expression
NON-PERTURBATIVE EXTENSION

\[ T(s, t, u; \{ \mu_d \}) = \sum_{p, q} E_{(p,q)}(\{ \mu_d \}) \sigma_2^p \sigma_3^q \]

\( \sim s^{2p+3q} + \ldots \)

\( E_{d+1}(\mathbb{Z}) \) invariant functions

e.g. \( SL(2, \mathbb{Z}) \) duality - modulus \( \Omega \)

- Nonlinear supersymmetry + \( SL(2, \mathbb{Z}) \) duality lead to Laplace equations with solutions:

\( \mathcal{R}^4 \quad E_{(0,0)}(\Omega) = E_{\frac{3}{2}}(\Omega) \quad \text{NON-REnormalisation beyond 1 loop} \)

\( \frac{1}{2} \) - BPS

\( d^4 \mathcal{R}^4 \quad E_{(1,0)}(\Omega) = E_{\frac{5}{2}}(\Omega) \quad \text{NON-REnormalisation beyond 2 loops} \)

\( \frac{1}{4} \) - BPS

Eisenstein series \( E_s(\Omega) \) has two power-behaved terms \( \Omega_2^s, \Omega_2^{1-s} \)

perturbative tree and (s-1/2)-loop

\( d^6 \mathcal{R}^4 \quad E_{(0,1)}(\Omega) \) not an Eisenstein series \( \text{NON-REnormalisation beyond 3 loops} \)

\( \frac{1}{8} \) - BPS

- All power-behaved terms agree precisely with explicit perturbative string calculations.

- Generalisations to higher-rank groups involve \textbf{maximal parabolic Langlands Eisenstein series.}

Toroidal compactifications

- Correct \( \frac{1}{2} \)-BPS and \( \frac{1}{4} \)-BPS instanton orbits – correspond to all the expected wrapped branes.
HAPPY BIRTHDAY, NATI