# Some Tools for Exploring Supersymmetric RG Flows


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Work in Progress with G. Festuccia and M. del Zotto

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# Quantum Field Theory and Supersymmetry

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Supersymmetric QFTs can display rich, non-conformal dynamics. They also have some protected (BPS) quantities that can be analyzed exactly. Example: superpotential $W(\Phi)$, holomorphic in all (background) chiral superfields, including couplings [Seiberg]. In favorable situations, tracking the protected quantities along the RG flow can give a good picture of the dynamics.

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$$\mathcal{M} = T^d, \quad Z_{\mathcal{M}} = \mathrm{Tr}_{\mathcal{H}}(-1)^F \ , \quad \mathcal{H} = \text{states on } T^{d-1} \times \mathbb{R}_{\text{time}}$$

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- Naturally defined in any (non-conformal) SUSY theory.
- Counts (with sign) the SUSY vacua on $T^{d-1}$: index.
- Varying parameters (RG flow) typically does not change the answer, but vacua can pair up and acquire positive energy.
- Subtle wall-crossing phenomena, sometimes ill defined.

# Supersymmetric Indices (cont.)

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Just as the index on $\mathcal{M} = T^d$, indices on $\mathcal{M} = S^{d-1} \times S^1$ can sometimes be defined for **non-conformal** supersymmetric theories:

- When can this be done, how to preserve SUSY (not obvious)?
- Not canonical: additional choices, parameters in curved space.
- Does the index depend on them? What does it count?
- Only non-conformal indices can be used to explore RG flows.

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The coupling of the QFT to background supergravity proceeds via the flat-space stress tensor $T_{\mu\nu}$, and its superpartners $\mathcal{J}_B^i$ and $\mathcal{J}_F^i$,

$$\Delta\mathscr{L} = -\frac{1}{2}\Delta g^{\mu\nu} T_{\mu\nu} + \sum_i \left(\mathcal{B}_B^i \mathcal{J}_B^i + \mathcal{B}_F^i \mathcal{J}_F^i\right) + (\text{seagull terms})$$

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- Typically, activate bosons $g_{\mu\nu}, \mathcal{B}_B^i$ and set fermions $\mathcal{B}_F^i = 0$.
- A supercharge $Q$ exists if

$$\delta_Q \mathcal{B}_F^i = 0 \quad \supset \quad \nabla_\mu \zeta + f\left(g_{\mu\nu}, \mathcal{B}_B^i\right)\zeta = 0\ .$$

These equations determine all SUSY backgrounds $(g_{\mu\nu}, \mathcal{B}_B^i)$.

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  These equations determine all SUSY backgrounds $(g_{\mu\nu}, \mathcal{B}^i_B)$.
- Activating only $g_{\mu\nu}$ is typically not enough to preserve SUSY ($[Q, T_{\mu\nu}] \neq 0$), or to specify the background (different $\mathcal{B}^i_B$).
- SUSY algebra, Lagrangians on $\mathcal{M}$ follow from supergravity.

# An $\mathcal{N} = 1$ Index in 4d

There are unitary $\mathcal{N} = 1$ theories on $S^3_\ell \times \mathbb{R}_{\text{time}}$ [Sen, Römelsberger, Festuccia-Seiberg]. The SUSY algebra is deformed to $\mathfrak{su}(2|1)$:

$$\left\{Q^{\dagger\alpha}, Q_\beta\right\} = \delta^\alpha{}_\beta\left(H + \frac{1}{\ell}R\right) + \frac{2}{\ell}J^\alpha{}_\beta\ , \qquad \{Q_\alpha, Q_\beta\} = 0$$

$\mathfrak{su}(2) \times \mathfrak{u}(1)$ subalgebra $=$ (left or right $SU(2)$ isometries of $S^3$) $\times$ (unbroken $U(1)_R$-symmetry). The Hamiltonian $H$ is central.

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- We can choose any $U(1)_R$ current $j^{(R)}_\mu$ in flat space. The couplings on $S^3$ explicitly depend on this choice.
- The coupling $V^\mu X_\mu$ only exists in non-conformal theories; it is crucial for preserving supersymmetry.
- In an SCFT: $\mathfrak{su}(2|1)\subset$ superconformal algebra, with $Q^{\dagger\alpha}\sim S^\alpha$ and $H \sim D + \frac{1}{2}R$

# An $\mathcal{N} = 1$ Index in 4d (cont.)

$\mathfrak{su}(2|1)$ unitarity bounds: $E\ell \geq 2j + 2 - r$ or $E\ell = -r$ $(j = 0)$.
Count short multiplets (modulo recombination) using an index:

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Everything is well defined as long as the spectrum of $H$ is discrete.

# Is $\mathcal{N} = 2$ Harder Than $\mathcal{N} = 1$?

When the $R$-charge $r$ of a scalar $\phi$ vanishes, the spectrum of $H$ is continuous, because the $r$-dependent curvature coupling vanishes:

$$\mathscr{L} \quad \supset \quad \partial^\mu \overline{\phi} \partial_\mu \phi + \frac{f(r)}{\ell^2} |\phi|^2 \; , \qquad f(r = 0) = 0$$

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This problem naturally arises in $\mathcal{N}=2$ theories:

- $\mathcal{N}=2$ SCFTs have $SU(2)_R \times U(1)_R$ symmetry, which can be used to define a well-behaved index [Kinney et. al.].
- Non-conformal $\mathcal{N}=2$ theories often preserve the $SU(2)_R$ symmetry, but typically not the $U(1)_R$ symmetry.
- Vector-multiplet scalars $\phi$ are neutral under $SU(2)_R$, i.e. $\mathcal{I}(q)$ does not exist in non-conformal $\mathcal{N}=2$ theories with vectors.

We expect that $\mathcal{N}=2$ supersymmetry allows us to do better!

# Searching for a Non-Conformal $\mathcal{N} = 2$ Index

As a guide, we start with an SCFT, but avoid certain generators:

- Conformally map the theory to $S^3_\ell \times \mathbb{R}$.
- Look for a subalgebra of the superconformal algebra that only includes genuine isometries of $S^3_\ell \times \mathbb{R}$ and $SU(2)_R$.

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The largest such subalgebra is another, but **different** $\mathfrak{su}(2|1)$:

$$\left\{\mathcal{Q}^{\dagger\alpha}, \mathcal{Q}_\beta\right\} = \delta^\alpha{}_\beta \left(H + \frac{1}{\ell} R_3\right) + \frac{2}{\ell} J^\alpha{}_\beta \ , \qquad \left\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\right\} = 0$$

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Short $\mathfrak{su}(2|1) \subset$ superconformal multiplets satisfy $\Delta = j_{\text{diag.}} + 2R_3$. The $\mathfrak{su}(2|1)$ index $\mathcal{I}(q)$ is the Schur limit of the $\mathcal{N} = 2$ superconformal index [Gadde-Rastelli-Razamat-Yan].

# Supergravity Background

We would like to construct non-conformal $\mathcal{N} = 2$ theories on $S^3 \times \mathbb{R}_{\text{time}}$ that realize this diagonal $\mathfrak{su}(2|1)$ SUSY algebra.

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Use background supergravity formalism of [Festuccia-Seiberg]:

1.) Choose a stress-tensor multiplet in flat space. Essentially all interesting $\mathcal{N} = 2$ theories with an $SU(2)_R$ symmetry have a distinguished stress-tensor multiplet discovered by [Sohnius]:

$$\mathcal{T} \;\to\; \psi^i_\alpha \;\to\; W_{[\mu\nu]}, R^{(ij)}_\mu, r_\mu \;\to\; S^i_{\mu\alpha} \;\to\; T_{\mu\nu} \qquad \text{(real)}$$

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2.) Need a background supergravity field for each operator:

$$\Delta\mathscr{L} = \mathcal{J}_{\mathcal{T}}\mathcal{T} + \mathcal{J}_W^{[\mu\nu]} W_{[\mu\nu]} + V^{\mu(ij)} R_{\mu(ij)} + A^\mu r_\mu - \frac{1}{2}\Delta g^{\mu\nu} T_{\mu\nu}$$
$$+ \mathcal{J}_X^{(ij)} X_{(ij)} + C^\mu z_\mu + \mathcal{J}_K K + \text{(c.c.)} + \text{(fermions)}$$

Note: $C_\mu$ is the dimensionless central-charge gauge field.

# Supergravity Background (cont.)

3.) Solve the SUSY conditions $\delta_Q(\text{SUGRA fermions}) = 0$. We found a solution with a round metric on $S^3_\ell$ of radius $\ell$,

$$ds^2 = -dt^2 + \ell^2 \left( d\theta^2 + \sin^2 \theta d\Omega_2 \right) \ , \quad 0 \leq \theta \leq \pi$$

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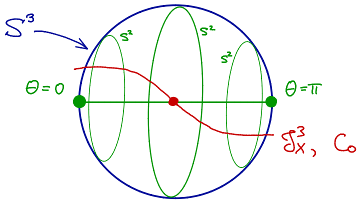

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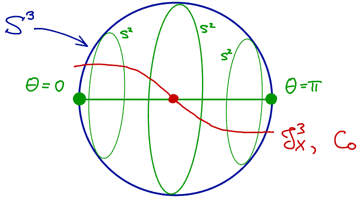


Some other fields break $SO(4) \to SU(2)_{\text{diag}}$, $SU(2)_R \to R_3$,

$$\mathcal{J}^3_X \sim \frac{\zeta}{\ell}\cos\theta \ , \qquad C_\mu dx^\mu \sim \zeta \cos\theta dt \qquad \text{(decouple in SCFT)}$$

$$\mathcal{J}_{\mathcal{T}} \sim \frac{1}{\ell^2} \ , \qquad V^3_\mu dx^\mu \sim \frac{1}{\ell} dt \ , \qquad \mathcal{J}_K = \zeta^2 = \text{constant phase}$$

# Supergravity Background (cont.)

In fact, there are four supercharges that give the desired $\mathfrak{su}(2|1)$.
We can analyze $\mathcal{N} = 2$ theories on this $S^3 \times \mathbb{R}_{\text{time}}$ background:

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- In non-conformal theories, the background fields $\mathcal{J}_X^3, C_0$ that break $SO(4)$, $SU(2)_R$ lead to **position-dependent mass terms and derivative couplings**.

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- Also possible to compute $\mathcal{I}(q)$ in the IR $\Rightarrow$ make contact with recent conjectures of [Iqbal-Vafa, Cordova-Shao + Gaiotto, ...] relating $\mathcal{I}(q)$ to BPS particles on the Coulomb branch.

# Comments on (Non-) Decoupling

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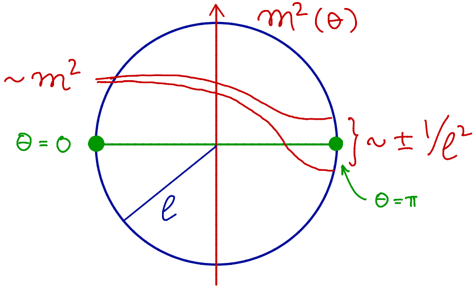
**This is inconsistent with naive decoupling:** as $m \to \infty$, the flat-space theory becomes trivial, with $\mathcal{I}(q) = 1$. On $S^3_\ell \times \mathbb{R}_{\mathsf{time}}$, we expect non-vacuum states to have energy $E \sim m \to \infty$.

# Comments on (Non-) Decoupling (cont.)

In fact, the position-dependent background fields lead to a non-trivial mass function $m^2(\theta)$ for some scalar modes, e.g.




- Near the poles, $m^2(\theta)$ can become very small. This leads to localized modes with $E \sim \frac{1}{\ell} \ll m$, which do not decouple.
- In natural units, the background fields are very strong:

$$m^2(\theta) \supset -m^2 C_0^2 \sim C_{\text{phys.}}^2 \,, \qquad C_0 \sim \cos\theta \,,$$

# Inserting BPS Line Operators

At the poles of $S^3$ the $S^2$ shrinks, $\mathfrak{su}(2|1)$ contracts to a subalgebra of flat-space SUSY. It is the algebra preserved by a BPS particle with central charge $Z \parallel \zeta$ (or its antiparticle).

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This algebra is also preserved by certain $\frac{1}{2}$-BPS line operators, studied by [Gaiotto-Moore-Neitzke, ...]. They are:

- Supported on straight lines $L$.
- Invariant under the maximal unbroken $SU(2)_R \times SU(2)_{\text{rot.}}$

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- Supported on straight lines $L$.
- Invariant under the maximal unbroken $SU(2)_R \times SU(2)_{\text{rot.}}$

Example: a Wilson line of charge $q$ for a $U(1)$ gauge field $A$,

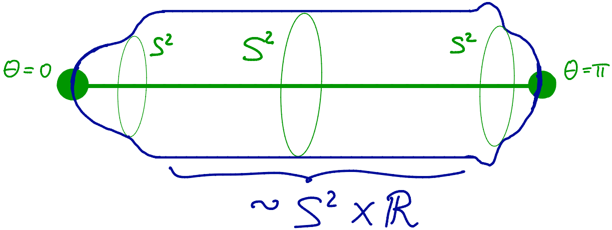
$$W_q = \exp\left(iq \int_L (A + \frac{i}{2\zeta}\phi - \frac{i\zeta}{2}\overline{\phi})\right)$$

These line defects can be inserted into our $S^3_\ell \times \mathbb{R}_{\text{time}}$ background, if we place them at the poles and along time.

# Deforming the Sphere

The background admits a family of deformations where the radius of $S^2$ is any bounded function $f(\theta)$ on the interval $0 \leq \theta \leq \pi$,

$$ds^2 = -dt^2 + \ell^2 \left( d\theta^2 + f(\theta)^2 d\Omega_2 \right)$$



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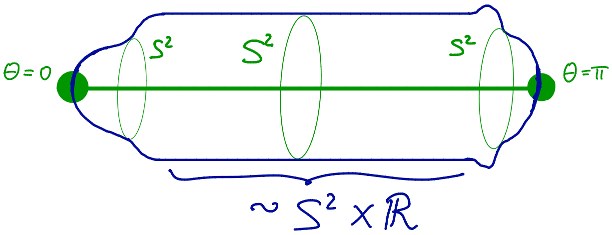


Any choice of $f(\theta)$ preserves the full $\mathfrak{su}(2|1)$ symmetry – again the indx $\mathcal{I}(q)$ is unchanged. If $f(\theta) = \text{const.}$ the geometry is $S^2 \times \mathbb{R}^{1,1}$ and the SUSY algebra enhances to $\mathfrak{su}(2|2)$. Theories with this algebra were studied by [Itzhaki-Kutasov-Seiberg, Lin-Maldacena,...].

# A State-Operator Map for BPS Lines

The unitarity bounds of $\mathfrak{su}(2|2)$ show that the theory on $S^2 \times \mathbb{R}^{1,1}$ must be fully gapped. It can have multiple isolated vacua that are invariant under $SU(2)_R \times SU(2)_{\text{rot.}}$. They cannot be lifted by any continuous parameter variations, including RG flow (very rigid).

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Example: free $U(1)$ gauge theory on $S^2 \times \mathbb{R}^{1,1}$ leads to axion electrodynamics on $\mathbb{R}^{1,1}$, with vacua labeled by any integer $q \in \mathbb{Z}$:

$$\mathscr{L}_{2d} \sim F_{01}^2 + (\partial\varphi)^2 + \varphi F_{01} \;, \qquad \langle \varphi \rangle = (\text{const.}) \times q$$

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The correspondence explains many observed features of these BPS defects, e.g. the one-to-one map between UV lines and IR lines on the Coulomb branch [Gaiotto-Moore-Neitzke, Cordova-Neitzke,...].

# Conclusions

- We defined a new $S^3 \times S^1$ index $\mathcal{I}(q)$ for non-conformal $\mathcal{N} = 2$ theories – generalizes the superconformal Schur index.
- In both cases, supergravity background fields and an $\mathfrak{su}(2|1)$ non-renormalization theorem played a crucial role.
- In asymptotically free or conformal gauge theories, $\mathcal{I}(q)$ can be computed using a simple matrix model (UV).
- $\mathcal{I}(q)$ is independent of mass deformations – naively violates decoupling of heavy states. No paradox: the background fields are very strong and can make some massive states light.
- Goal: compute $\mathcal{I}(q)$ in the IR, or perhaps some intermediate description that includes massive (BPS) particles.
- $\mathcal{I}(q)$ can be decorated with $\frac{1}{2}$-BPS line defects. They are in one-to-one correspondence with the massive vacua of the theory on an $S^2 \times \mathbb{R}^{1,1}$ background with $\mathfrak{su}(2|2)$ symmetry.

**Thank You for Your Attention**

**and**

# Happy Birthday Nati!