Feynman Diagrams In String Theory

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I will aim to explain the minimum about string perturbation theory that every quantum physicist should know.
When we look at a Feynman diagram

we assign a propagator \( \frac{1}{p^2 + m^2} \) to each line.

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It is convenient to think of the Feynman graph $\Gamma$ as a singular 1-manifold, with singularities at the vertices. What it means to assign the length parameters $t_i$ to the graph is just that we have a Riemannian metric on $\Gamma$. Up to diffeomorphisms of the internal lines in $\Gamma$, the only invariants of such a metric are the lengths of the line segments, that is, the $t_i$. 
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we want a delta function

\[(2\pi)^4 \delta^4(p_1 + p_2 + p_3).\]
We can conveniently get that delta function from
\[
\int d^4x \exp(i \sum_i p_i \cdot x) = (2\pi)^4 \delta^4(\sum_i p_i).
\]

So we assign a spatial coordinate to each vertex, and we write the propagator in position space as
\[
G(x, y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip \cdot (x-y)}}{p^2 + m^2}
\]
\[
= \int_0^\infty dt \frac{d^4p}{(2\pi)^4} \exp(ip \cdot x - t(p^2 + m^2)).
\]
So now we have a slightly new way to interpret a Feynman diagram:

\[
G(x, y; t) = \int d^4 p \left( \frac{2\pi}{4} \right)^4 e^{ip \cdot (x - y) - t(p^2 + m^2)}.
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So now we have a slightly new way to interpret a Feynman diagram:

We integrate over a position parameter $x$ for each vertex, and a length parameter $t$ for each line. In addition, each line has a factor

$$G(x, y; t) = \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x-y)} - t(p^2 + m^2).$$
However, in addition to inventing Feynman diagrams, Feynman also taught us how to interpret the function

\[ G(x, y; t) = \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x - y) - t(p^2 + m^2)}. \]

We think about a non relativistic point particle with Hamiltonian

\[ H = p^2 + m^2, \quad p = -i \frac{d}{dx}. \]

The action for such a particle is

\[ I = \int dt \left( \left( \frac{dx}{dt} \right)^2 + m^2 \right), \]

which is just the action for a non relativistic point particle with \( 2m = 1 \), and a constant \( m^2 \) added to the Lagrangian density.
According to Feynman, $G(x, y; t)$ can be obtained as an integral over all paths $X(t')$ for which $X(0) = y$, $X(t) = x$, or in other words all paths by which the particle travels from $y$ to $x$ in time $t$:

$$G(x, y; t) = \int DX(t') \exp \left( - \int_0^t dt' \left( \sum_i \left( \frac{dX_i}{dt'} \right)^2 + m^2 \right) \right).$$
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This is the basic Feynman path integral of non-relativistic quantum mechanics, which you can read about in for example the book by Feynman and Hibbs.
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In short, to evaluate the amplitude associated to a graph $\Gamma$, we integrate over (1) all possible metrics on $\Gamma$, modulo diffeomorphisms of $\Gamma$, and (2) all possible maps of $\Gamma$ into space-time.

This amounts to a version of 1-dimensional General Relativity, with the fields being a metric on $\Gamma$ and a map from $\Gamma$ to space-time.

If we write $h$ for the $1 \times 1$ metric tensor of $\Gamma$, and $g$ for the $d \times d$ metric tensor of space-time (for example $d = 4$), then the action in this one-dimensional General Relativity is

$$I = \int_{\Gamma} ds \sqrt{h} \left( h - \sum_i g_{ij} dX_i ds + m^2 \right).$$

Some points to note:

1. There is no purely Einstein action $\int_{\Gamma} ds \sqrt{\det h} R$, because there is no curvature $R$ in 1 dimension.
2. We can go to a gauge in which $h = 1$ and the integral over metrics reduces to an integral over the Schwinger parameters $t^i$.
3. Previously, we took the space-time metric to be just $g_{ij}(x) = \delta_{ij}$, with space-time being flat, but we do not have to assume this.
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An important point is that the integral over each Schwinger parameter $t$ has two ends. There is $t \to \infty$ which generates the pole of the propagator:

$$\int_{\Lambda}^{\infty} dt \exp(-t(p^2 + m^2)) \sim \frac{1}{p^2 + m^2}.$$
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We do not want to do without this region since the physical interpretation of quantum field theory depends crucially on the pole of the propagator!
The other end of the $t$ integral is responsible for the fact that the propagator is singular at short distances:

$$\int_0^\Lambda dt \int \frac{d^dp}{(2\pi)^d} e^{ip\cdot(x-y)-t(p^2+m^2)} \sim \frac{1}{|x-y|^{d-2}}.$$  

This singular short distance behavior of the propagator comes completely from the small $t$ part of the integral.
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I will call such a 2-manifold “the string worldsheet.”
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Whatever particles there are going to be represent different states
of vibration of one basic string. Also there are not any vertices in
the string worldsheet so we do not have the freedom to tell the
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$$ h = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} $$

has 3 independent components, but a diffeomorphism generator $x^i \rightarrow x^i + \epsilon^i(x)$, $i = 1, 2$ only depends on 2 functions, not enough to gauge fix the three components of $h$. One way to proceed is to accept the fact that we will have to do a path integral over this field, and try to make sense of it. It is not easy to go down that road, and it turns out to be a longer way to get to a destination at which we can arrive more directly (and incisively) by another route.
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$h_{ij} \sim e^{2 \sigma} h_{ij}$ for any function $\sigma$. 
The other route is as follows. We get something nice if we impose an extra symmetry that eliminates 1 component of $h$. We do that by requiring conformal or Weyl invariance

$$h_{ij} \cong h_{ij} e^{2\sigma}$$

for any function $\sigma$. 
A 19th century result says that, up to diffeomorphisms and conformal or Weyl transformations, a two-manifold $\Sigma$ only depends on finitely many parameters:

The moduli $\tau_i$ of the Riemann surface are rather analogous to the proper times $t_i$ of the edges of a corresponding Feynman graph, except that they are complex while the $t_i$ are real.
More concretely,

\[ \tau_i = s_i + it_i \]

where \( t_i \) is the “length” of a tube and \( s_i \) is a “twist angle.”
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Just as in 1 dimension, we are going to add “matter” to our 2-dimensional gravity theory. Since we are trying to use conformal invariance to improve the analogy between 2 dimensions and 1 dimension, by reducing the integral over 2-dimensional metrics to an integral over finitely many parameters $\tau_i$, the “matter” part of the action has to be conformally invariant. For instance, in 2 dimensions, the usual action for massless scalar fields is conformally invariant:

$$I = \int_\Sigma d^2\sigma \sqrt{\text{det}g} \left( g^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^J G_{IJ}(X) \right).$$

The $X$’s describe a map from the two-manifold $\Sigma$ to a spacetime $M$, which can have $D$ dimensions and which I've endowed with a metric tensor $G_{IJ}(X)$. 
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is a total derivative (topological invariant). So just as in ordinary quantum field theory, we basically have to only consider the action for the matter fields.
To develop the theory, we are supposed to (i) for a fixed $\Sigma$, do a Feynman path integral over the fields $X = (X^1, \ldots, X^D)$, and (ii) then integrate over the moduli $\tau_1, \tau_2, \ldots$ and sum over all topological choices for $\Sigma$. The last step is the analog of summing over all Feynman graphs in ordinary field theory.
To understand why the ultraviolet divergences go away, let us consider the basic 1-loop contribution to the vacuum energy.
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$$\frac{1}{\sqrt{\det(-\nabla^2 + m^2)}} = \exp\left(-\frac{1}{2} \mathrm{Tr} \log(-\nabla^2 + m^2)\right).$$

This means that the 1-loop contribution to the effective action is

$$I^* = \frac{1}{2} \mathrm{Tr} \log(-\nabla^2 + m^2) = \frac{1}{2} \int_0^\infty \frac{dt}{t} \exp(-tH)$$

where $H = p^2 + m^2 = -\nabla^2 + m^2$. 
The divergence is at $t = 0$:
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It is very instructive in field theory to see that the factor $1/2t$ comes from the symmetries of the graph.
This integral diverges for $t \to 0$. What we are about to see is that the string theory problem is similar, except that the integral only goes over $t \geq 1$, so there will be no divergence.
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(I will take a shortcut and assume the torus is conformally equivalent to a rectangle with opposite sides glued together rather than a more general parallelogram. This doesn’t affect the conclusion, but it shortens the explanation.)
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In field theory, the circle has a circumference $t$. In string theory, a rectangle with a flat metric would have two parameters $s$ and $s'$:

We could view the rectangle as describing a string of circumference $s$ propagating through a proper time $s'$ or a string of circumference $s'$ propagating through a proper time $s$. Either picture is correct. In any event, in string theory, because of conformal invariance, only the ratio $s/s'$ (or its inverse $s'/s$) is meaningful.
The string theory formula will reduce approximately to field theory if, say, $s \gg s'$. Then we can think in terms of a string of circumference $s'$ propagating for a proper time $s$. Because of conformal invariance we can set $s' = 1$ and identify $s$ with the proper time $t$ of a field theory: $t = s/s' = s$. So the integral that in field theory is an integral over the proper time $t$ is in string theory replaced by an integral over the ratio $s/s'$. 
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Because of this symmetry, we are free to restrict the integration region to $s \geq s'$. In other words, if we set $t = s/s'$, we are restricted to $t \geq 1$. 
In short, in field theory we integrate over $t$ from $0$ to $\infty$ and we typically find ultraviolet divergences for $t \to 0$.

(Depending on the theory, we may find infrared divergences for $t \to \infty$.)

In string theory, we integrate over $t$ from $1$ to $\infty$. There is no ultraviolet divergence since the integral begins at $1$. (Depending on the theory, there may be an infrared divergence for $t \to \infty$.)
In short, in field theory we integrate over $t$ from 0 to $\infty$ and we typically find ultraviolet divergences for $t \to 0$. (Depending on the theory, we may find infrared divergences for $t \to \infty$.)
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What is true but much less trivial is the following statement: The infrared behavior of string theory matches the infrared behavior of a field theory with appropriate light particles and interactions.
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By now I have done what I said at the beginning, but the organizers are so generous with the time that I will explain one more thing: why string theory describes gravity in the target spacetime, which I call $M$. This isn’t up to us; it happens automatically. To explain this takes a couple of steps.
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In other words, the fields are the vibrational states of a string.
If we change the metric $G_{IJ}$ of $M$ a little bit, by $G \rightarrow G + \delta G$, the action changes by

$$\delta l = \int_{\Sigma} d^2\sigma \sqrt{\text{det} h} \left( h^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^J \delta G_{IJ}(X) \right).$$
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This causes an operator insertion on the Riemann surface somewhere (we have to integrate over where):

For the case that the change in the action comes from a change in the metric in spacetime, the operator is $O = h^{\alpha\beta} \partial_{\alpha} X^I \partial_{\beta} X^J \delta G_{IJ}(X)$. 
By the magic of conformal invariance, the point where we inserted the operator can be projected to infinity:
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So it corresponds to one of the external fields – and if we shift the expectation value of that field, this amounts to shifting the metric $G_{IJ}$ in spacetime.
So the dynamics of gravity in spacetime is part of what string theory describes.
So the dynamics of gravity in spacetime is part of what string theory describes. A shift in the metric of spacetime is a shift in the expectation value of one of the string fields; or differently put, one of the string fields is the spacetime metric.
Going down this road (and incorporating spacetime supersymmetry to avoid some infrared problems), one arrives at a systematic way to calculate quantum processes involving gravitons, free of the ultraviolet divergences that one gets if one tries to quantize Einstein’s theory directly.
Going down this road (and incorporating spacetime supersymmetry to avoid some infrared problems), one arrives at a systematic way to calculate quantum processes involving gravitons, free of the ultraviolet divergences that one gets if one tries to quantize Einstein’s theory directly. The ultraviolet divergences are absent because two-dimensional conformal invariance completely eliminates the ultraviolet region from the Feynman diagrams.