1. The stability of planetary systems

To understand the formation and evolution of exoplanet systems, we would like to have empirical theoretical tools for predicting whether a given planetary system is stable—or, more precisely, what is its lifetime before some catastrophic event such as a collision or ejection—without having to integrate the planetary orbits for the lifetime of the host star. There are two discouraging lessons from studies of the stability of the solar system that we should bear in mind. First, small changes in the initial conditions or system parameters can lead to large changes in the lifetime. Second, chaos does not necessarily imply instability: in the solar system the likely lifetime is at least 1000 times longer than the Liapunov time.

One approach to this problem is to consider planetary systems in which both the planet mass and the separation between adjacent planets are very small. In examining this limit a useful parameter is the Hill radius. Consider a planet of mass \( m \) in a circular orbit of radius \( r \) around a host star of mass \( M \). A test particle also circles the host star, in an orbit of radius \( r + x \) with \( 0 < x \ll r \). The planet orbits with angular speed \( n = (GM/r^3)^{1/2} \) which is slightly faster than the angular speed of the test particle, \( [GM/(r + x)^3]^{1/2} \). As the planet overtakes the test particle the maximum force per unit mass that it exerts on the test particle is \( F \sim Gm/x^2 \) and this force lasts for a duration \( \Delta t \sim n^{-1} \); thus the velocity change is \( \sim F\Delta t \sim Gm/(x^2n) \). The change is as large as the initial relative velocity \( v \sim nx \) if \( F\Delta t \gtrsim v \) which implies \( x^3 \lesssim Gm/n^2 \equiv r^3(m/M), \) or \( x \lesssim r_H \) where

\[
    r_H = r \left( \frac{m}{3M} \right)^{1/3} \tag{1}
\]

is the Hill radius. The Hill radius appears in other contexts in astrophysics, such as the tidal or limiting radius of star clusters, the Roche limit for solid satellites, and the location of the Lagrange points (the factor of 3 in this formula ensures that the Hill radius is equal to the distance to the collinear Lagrange points).

For two planets of comparable mass, \( m_1, m_2 \ll M \), the analog is the mutual Hill radius, defined as

\[
    r_H = \frac{r_1 + r_2}{2} \left( \frac{m_1 + m_2}{3M} \right)^{1/3} \tag{2}
\]
The use of the arithmetic mean of the radii (rather than, say, the geometric mean) is arbitrary but the choice of what mean to use makes very little difference since the mutual Hill radius is only dynamically important when \( r_1 \) and \( r_2 \) are similar, since we have assumed that \( m_1, m_2 \ll M \). In contrast, we use the sum of the masses (rather than, say, \( (m_1/3M)^{1/3} + (m_2/3M)^{1/3} \)) because the dynamics of two small bodies during a close encounter depends on the masses only through their sum (Petit & Hénon 1986, Icarus 66, 536).

A system of two small planets on nearby circular, coplanar orbits is stable for all time if \(|r_1 - r_2| > 3.46r_H\) (the standard reference is Gladman 1993, Icarus 106, 247, although forms of this criterion were derived earlier). Note that the scaling \(|r_1 - r_2| > \text{const} \times (m/M)^{1/3}\) does not capture all of the interesting dynamics of two bodies on nearby orbits; for example, the criterion that the orbits are regular rather than chaotic is \(|r_1 - r_2| > \text{const} \times (m/M)^{2/7}\) (Wisdom 1980, AJ 85, 1122).

For systems of three or more planets there are no exact stability criteria. Moreover such systems are not stable for all time—the instability time becomes very large if the planets have small masses or large separations but such systems always eventually exhibit instability. Numerical orbit integrations imply that a system of three or more planets on nearly circular, coplanar orbits is stable for at least \( N \) orbital periods if the separation between adjacent planets is \(|r_{i+1} - r_i| > K(N)r_H\) where, for example, \( K(10^{10}) \approx 9–12 \) (e.g., Smith & Lissauer 2009, Icarus 201, 381).

In summary, a system of planets on nearly circular, coplanar orbits is expected to be stable for the stellar lifetime if the separation of adjacent planets exceeds \( \sim 10 \) mutual Hill radii. To do better than this crude criterion, our only option is to integrate the planetary orbits.

### 2. Hamiltonian perturbation theory

#### 2.1. The disturbing function

The dynamics of any one body in the N-body problem can be described by a Hamiltonian \( H(q, p) = H_K(q, p) + H_1(q, p, t) \) where \( H_K = \frac{1}{2}|p|^2 - G M / |q| \) is the Kepler Hamiltonian and \( H_1 \) represents the perturbing gravitational potential from the other planets, or from passing stars, the equatorial bulge of the host star, etc. We are interested mainly in near-Keplerian problems where \(|H_1| \ll |H_K|\). The goal of perturbation theory in celestial mechanics is to find solutions for motion in this Hamiltonian that are valid when \( H_1 \) is “small enough”—in fact it is for this problem that perturbation theory was invented. The goal of this section is to persuade you that Hamiltonian perturbation theory is straightforward to apply in simple
systems, and provides answers and insight that cannot easily be obtained in any other way.

In a Hamiltonian system we can replace \((q, p)\) with any other canonical coordinates and momenta. In particular we may use the Delaunay elements\(^1\) to obtain

\[
H = -\left(\frac{\Theta M}{2L^2}\right)^2 + H_1(L, G, H, \ell, \omega, \Omega, t). \tag{3}
\]

The function \(H_1\) is called the \textbf{disturbing function} (unfortunately in many celestial mechanics books this name is used to refer to \(-H_1\)). Deriving \(H_1\) is much harder in Delaunay elements than in Cartesian coordinates (for example, it depends on all six Delaunay elements but only three Cartesian coordinates) but the advantage of working in Delaunay elements is so great that this effort is usually worthwhile.

Some restrictions on the form of \(H_1\) can be derived from general considerations. If we replace \(\ell\) by \(\ell + 2\pi\) the position of the particle is unchanged, so \(H_1\) is unchanged. Thus \(H_1\) must be a periodic function of \(\ell\) with period \(2\pi\). Similarly it must be a periodic function of \(\omega\) and \(\Omega\) with the same period. Thus it can be written as a Fourier series

\[
H_1 = \sum_{k,m,n} H_{k,m,n}(L, G, H, t) \cos[k\ell + m\omega + n\Omega + \phi_{k,m,n}(t)], \tag{4}
\]

where as usual the mean longitude \(\lambda = \ell + \omega + \Omega\) and the longitude of periapsis \(\varpi = \omega + \Omega\).

When the eccentricity is zero \((G = L)\), \(H_1\) cannot depend on the periapsis direction \(\varpi\). Similarly, when the inclination is zero \((H = G)\), \(H_1\) cannot depend on the nodal longitude \(\Omega\). More generally, \(H_1\) has to be an analytic function of the position and this restricts the possible forms of \(H_1\) to satisfy the d’Alembert property\(^2\)

\[
H_{k,m,n}(L, G, H, t) = O[(L - G)^{m-k/2}, (G - H)^{m-n/2}]. \tag{5}
\]

Now suppose that the perturbing potential is stationary in a rotating frame, \(\Phi[R, z, \phi - \phi_s(t)] = \sum_{m'} \Phi_{m'}(R, z) \cos m' \phi - \phi_s(t)\). If we rotate the origin of the azimuthal coordinate system by \(\psi\) then \(\phi\) and \(\phi_s\) both increase by \(\psi\) but the potential remains the same. Similarly,

\(^1\)As a reminder, the Delaunay momenta are \(L = (GMa)^{1/2}\), \(G = L(1 - e^2)^{1/2}\), \(H = G \cos i\) and the conjugate coordinates are the mean anomaly \(\ell\), the argument of periapsis \(\omega\), and the longitude of the ascending node \(\Omega\).

\(^2\)Similarly, if a smooth function \(f(x, y)\) of the Cartesian coordinates \(x\) and \(y\) is rewritten in polar coordinates \((r, \phi)\) then its expansion in powers of \(r\) will have the form \(c_0 + c_1 r \cos(\phi - \phi_1) + c_2 r^2 \cos 2(\phi - \phi_2) + \cdots\).
the longitude of the node $\Omega$ increases by $\psi$ but the elements $\ell$ and $\omega$ do not change. In order that $H_1$ does not change, we then require that $m = m'$ and $\phi_{kmn} = -m\phi_s(t)$.

Thousands of pages have been written about the disturbing function in the gravitational N-body problem and its approximate forms for nearly circular, nearly coplanar orbits like those found in the solar system. Here we will discuss only three specific problems to give some flavor of these results and how the disturbing function is used in practice.

### 2.2. Satellite orbits

We first analyze some of the properties of the orbit of a satellite around a planet. The external gravitational potential from a planet with an axis of symmetry can be written

$$
\Phi(r) = -\frac{\Theta m}{r} \left[ 1 - \sum_{n=2}^{\infty} \frac{J_n R^n}{r^n} P_n(\cos \theta) \right]
$$

where $m$ and $R$ are the mass and mean radius of the planet, $\theta$ is the polar angle from the symmetry axis, $P_n$ is a Legendre polynomial, and $J_n$ is a dimensionless multipole moment. There is no $n = 1$ component if the center of mass is chosen to be the origin of the coordinates. In most cases the dominant component is $n = 2$; since $P_2(\cos \theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2}$ we then have

$$
\Phi(r) = -\Theta m \frac{J_2 R^2}{r^5} (3z^2 - r^2)
$$

where $z$ is the component of the radius vector along the symmetry axis. The corresponding disturbing function is

$$
H_1 = \frac{\Theta m J_2 R^2}{2r^5} (3z^2 - r^2). \quad (8)
$$

In converting this potential to a disturbing function of the form (4) we note that $\Phi$ has no explicit time dependence so $\phi_{kmn}$ is independent of time. Since the mean longitude $\lambda$ rotates through $2\pi$ in one orbital period, while $\varpi$ and $\Omega$ are constants for a Keplerian orbit, the terms in the disturbing function are of two types: short-period terms with $k \neq 0$ and secular terms with $k = 0$. Although the short-period terms are important for precise predictions of the location of the satellite at a given instant, the largest changes in its orbit will come from the secular terms since these act for a much longer time.

Restricting the disturbing function to its secular terms is equivalent to averaging over one orbit. Denoting this average by $\langle \cdot \rangle$ and evaluating $\langle z^2/r^5 \rangle$ and $\langle 1/r^3 \rangle$ using equations (50) and (43) of Lecture 1, we find

$$
H_{sec} = \langle H_1 \rangle = \frac{\Theta m J_2 R^2}{4a^3(1-e^2)^{3/2}} (3\sin^2 I - 2) = \frac{(\Theta m)^4 J_2 R^2}{4L^3G^3} (1 - 3H^2/G^2).
$$

(9)
With this disturbing function, most of the properties of the satellite orbit are easy to understand. For example, Hamilton’s equations imply that \( \frac{dL}{dt} = -\frac{\partial H}{\partial \ell} = 0 \) so the semi-major axis \( a = \frac{L^2}{(\mathcal{G} M)} \) is constant, except for small short-period variations. Similarly, \( \frac{dG}{dt} = -\frac{\partial H}{\partial \omega} = 0 \) so the angular momentum and eccentricity are constants except for short-period variations. Similarly, \( \frac{dH}{dt} = -\frac{\partial H}{\partial \Omega} = 0 \) so \( H \) is constant and the inclination \( I = \cos^{-1} \frac{H}{G} \) is fixed. The apsis direction or perigee \( \omega \) precesses at a rate

\[
\dot{\omega} = \frac{\partial H}{\partial G} = \frac{3(\mathcal{G} m)^4 J_2 R^2}{4 L^3 G^6} (5H^2 - G^2) = \frac{3(\mathcal{G} m)^{1/2} J_2 R^2}{4a^{7/2}(1 - e^2)^2} (5 \cos^2 I - 1).
\]

Thus the apsis direction of low-inclination orbits precesses forward; the apsis direction of high-inclination orbits precesses backward; and at the so-called critical inclination \( I = \cos^{-1} \sqrt{1/5} = 63.4^\circ \) the apsis does not precess.

The node precesses at a rate

\[
\dot{\Omega} = \frac{\partial H}{\partial H} = -\frac{3(\mathcal{G} m)^{1/2} J_2 R^2}{2a^{7/2}(1 - e^2)^2} \cos I,
\]

which is retrograde for all orbits with \( 0 \leq I < 90^\circ \).

### 2.3. Lidov-Kozai oscillations

Suppose that a planet of mass \( m \) orbits a host star of mass \( m_0 \), with \( m + m_0 = M \). Its position relative to the host star is \( \mathbf{x} \). There is a companion star of mass \( m_c \) with position \( \mathbf{x}_c \) relative to the host star, with \( |\mathbf{x}_c| \gg |\mathbf{x}| \). Expanding the heliocentric potential (eq. 37 of Lecture 2) in inverse powers of \( |\mathbf{x}_c| \),

\[
\Phi(\mathbf{x}) = -\frac{\mathcal{G} M}{x} - \frac{\mathcal{G} m_c}{x_c \sqrt{1 - 2 \mathbf{x} \cdot \mathbf{x}_c / x_c^2 + x^2 / x_c^2}} + \frac{\mathcal{G} m_c \mathbf{x} \cdot \mathbf{x}_c}{x_c^3} \\
= \frac{\mathcal{G} M}{x} - \frac{\mathcal{G} m_c}{x_c^3} \left[ \frac{1}{2x_c^2} + \mathbf{x} \cdot \mathbf{x}_c - \frac{1}{2} \frac{3(\mathbf{x} \cdot \mathbf{x}_c)^2}{2x_c^2} \right] + \frac{\mathcal{G} m_c \mathbf{x} \cdot \mathbf{x}_c}{x_c^3} + O(x^3 / x_c^4) \\
= \frac{\mathcal{G} M}{x} + \frac{\mathcal{G} m_c}{x_c^3} \left[ \frac{1}{2x_c^2} - \frac{3(\mathbf{x} \cdot \mathbf{x}_c)^2}{2x_c^2} \right] + O(x^3 / x_c^4);
\]

\[(12)\]

\( ^3 \)This is a property of the quadrupole potential that is not shared by other multipoles.
in the last line we have dropped an unimportant constant term $-\Theta m_c/x_c$. Dropping the terms of order $x^3/x_c^4$ gives us the **quadrupole approximation**, in which the disturbing function is

$$H_1 = \frac{\Theta m_c}{x_c^3} \left[ \frac{1}{2} x^2 - \frac{3}{2} (x \cdot x_c)^2 \right]. \quad (13)$$

Let $\hat{e}, \hat{u}, \hat{n}$ be unit vectors pointing respectively towards periapsis ($f = 0$), towards $f = \frac{1}{2}\pi$ in the orbital plane, and along the angular-momentum vector (thus $\hat{e} \times \hat{u} = \hat{n}$). Then $x = r(f)(\cos f \hat{e} + \sin f \hat{u})$ with a similar formula for $x_c$. As in the previous section, the largest effect on the orbit comes from the secular terms, which are obtained by averaging $H_1$ over the orbit. In this case we want to average over both the inner and outer orbits:

$$H_{\sec} = \langle H_1 \rangle = \Theta m_c \left[ \frac{1}{2} \langle r^2 \rangle \left( \frac{1}{r_c^3} \right) - \frac{3}{2} (\hat{e} \cdot \hat{e}_c)^2 \langle r^2 \cos^2 f \rangle \left( \frac{\cos^2 f_c}{r_c^3} \right) 
- \frac{3}{2} (\hat{e} \cdot \hat{u}_c)^2 \langle r^2 \cos^2 f \rangle \left( \frac{\sin^2 f_c}{r_c^3} \right) 
- \frac{3}{2} (\hat{u} \cdot \hat{u}_c)^2 \langle r^2 \sin^2 f \rangle \left( \frac{\cos^2 f_c}{r_c^3} \right) 
- \frac{3}{2} (\hat{u} \cdot \hat{n}_c)^2 \langle r^2 \sin^2 f \rangle \left( \frac{\sin^2 f_c}{r_c^3} \right) \right]. \quad (14)$$

Using equations (43) of Lecture 1, this simplifies to

$$H_{\sec} = \frac{\Theta m_c a^2}{4a_c^3(1 - e_c^2)^{3/2}} \left\{ 2 + 3e^2 - \frac{3}{2} [(\hat{e} \cdot \hat{e}_c)^2 + (\hat{e} \cdot \hat{u}_c)^2] (1 + 4e^2) 
- \frac{3}{2} [(\hat{u} \cdot \hat{e}_c)^2 + (\hat{u} \cdot \hat{u}_c)^2] (1 - e^2) \right\}. \quad (15)$$

Using the relation $(\hat{e} \cdot \hat{e}_c)^2 + (\hat{e} \cdot \hat{u}_c)^2 = 1$ and a similar relation for $\hat{u}$, this simplifies further to

$$H_{\sec} = \frac{\Theta m_c a^2}{4a_c^3(1 - e_c^2)^{3/2}} \left[ -1 - \frac{3}{2} e^2 + \frac{3}{2} (\hat{e} \cdot \hat{n}_c)^2 (1 + 4e^2) + \frac{3}{2} (\hat{u} \cdot \hat{n}_c)^2 (1 - e^2) \right]. \quad (16)$$

We choose the reference frame to be the orbital plane of the companion star. Then $\hat{e} \cdot \hat{n}_c = \sin I \sin \omega$ and $\hat{u} \cdot \hat{n}_c = \sin I \cos \omega$ and

$$H_{\sec} = \frac{\Theta m_c a^2}{8a_c^3(1 - e_c^2)^{3/2}} \left[ 1 - 6e^2 - 3(1 - e^2) \cos^2 I + 15e^2 \sin^2 I \sin^2 \omega \right] 
= \frac{m_c L^4}{8 \Theta M^2 a_c^3(1 - e_c^2)^{3/2}} \left[ \frac{6G^2}{L^2} - 5 - \frac{3H^2}{L^2} + 15 \left( 1 - \frac{G^2}{L^2} \right) \left( 1 - \frac{H^2}{G^2} \right) \sin^2 \omega \right]. \quad (17)$$

The secular Hamiltonian is independent of $\ell$ so the semi-major axis is constant, and independent of $\Omega$ so $H$ is constant. In contrast to the Hamiltonian (9), however, this Hamiltonian depends on the argument of periapsis $\omega$, so the conjugate variable $G$ (the angular momentum)
is not constant. This gives rise to the remarkable phenomenon of Lidov-Kozai oscillations, which we now describe.

Suppose that the initial planetary orbit is circular and inclined by $I_0$ to the reference plane (the orbital plane of the binary companion). Then $H = (GMa)^{1/2} \cos I_0$ and $H$ and $L$ are conserved, so conservation of $H_{sec}$ requires

$$2 - 2e^2 + 5 \sin^2 \omega (e^2 - \sin^2 I_0) = 0 \quad \text{or} \quad e = 0. \quad (18)$$

It is not hard to show that the first of these equations has a solution for $0 < e < 1$ if and only if $I_0 > \sin^{-1} \sqrt{2/5} = 39.2^\circ$. For larger initial inclinations the circular orbit may (and does) undergo oscillations to high eccentricity (Lidov–Kozai oscillations). The maximum eccentricity occurs at $\sin \omega = 1$ and is equal to

$$e_{\max} = \sqrt{\frac{5 \sin^2 I_0 - 2}{3}}; \quad (19)$$

as $I_0$ approaches $90^\circ$ the maximum eccentricity approaches unity (i.e., a collision orbit). Contours of equation (18 for $I_0 = 60^\circ$ are shown in Figure 1.

Among the remarkable properties of Lidov–Kozai oscillations are that (i) above the critical inclination of $39.2^\circ$, circular orbits are chaotic (because of additional terms due to octupole and higher moments of the tidal potential); (ii) the mass and distance of the companion star affect the period of the oscillations but not their amplitude. This means that in principle arbitrarily small external forces can lead to large changes in the orbit. In practice, when the perturber is sufficiently weak, either the oscillation period becomes longer than the age of the system, or other sources of precession—additional planets or general relativity—become stronger than the precession due to the companion, and this quenches the oscillations.

Note that there is an accidental degeneracy in the problem as formulated here: the Hamiltonian does not depend on the orientation of the apsis of the companion star. In more realistic treatments that include higher-order multipole moments this degeneracy is lifted, giving the dynamics an additional degree of freedom and thus more complicated behavior.

Fig. 1.— Lidov–Kozai oscillations of a planet that is initially in a circular orbit with inclination $I_0 = 60^\circ$ to the orbit of a distant companion star. The vertical axis is eccentricity and the horizontal axis is argument of periapsis. The contours are level surfaces of the energy (18).

2.4. Mean-motion resonances

Orbital resonances are ubiquitous in the solar system (Peale 1976, ARAA 14, 215). Some examples include the 3:2 resonance between the orbital periods of Neptune and Pluto and other Kuiper belt objects; the 2:1 resonance between the orbital periods of Saturn’s satellites Mimas and Tethys, and Dione and Enceladus; the co-orbital (1:1) resonance between Saturn’s satellites Janus and Epimetheus; the Kirkwood gaps in the asteroid belt and the gaps in Saturn’s rings; the three-body resonance between the mean motions of the first three Galilean satellites ($n_{\text{Io}} - 3n_{\text{Europa}} + 2n_{\text{Ganymede}} = 0$); Trojan asteroids; etc.

To keep the discussion as simple as possible, we focus on resonances between a planet on a circular orbit and a test particle orbiting on a near-circular orbit interior to the planet. We shall also assume that the orbits are coplanar. We denote the semi-major axis and mean motion of the planet’s orbit by $a_c$ and $n_c$, and its mass by $m_c$. According to the discussion
in §2.1 the disturbing function can be written

\[ H_1 = \sum_{k,m,n} H_{kmn}(L, G, H) \cos(k\ell + m\omega + n\Omega - mn_*t); \] (20)

we have chosen the origin of time so that the planet is at azimuthal angle \( \phi_c = 0 \) at \( t = 0 \).

The Hamiltonian is further restricted by the d’Alembert property (5). We are assuming that the satellite and test-particle orbits are coplanar so \( G = H \), which implies that the only non-zero terms have \( n = m \). We are also assuming that the eccentricity of the test particle is small, so \( H_{kmn} \sim e^{|m-k|} \). Thus the largest terms in the Hamiltonian have \( k = m \), so we may approximate

\[ H_1 = \sum_m A_m \cos[m(\ell + \omega + \Omega - n_*t)] = \sum_m H_m(L) \cos[m(\lambda - n_*t)]. \] (21)

Resonance occurs when the rate of change of the argument of the cosine is small, that is, when \( n \approx n_c \). In other words the mean motions of the test particle and the planet must be nearly the same; examples of this resonance (also called a 1:1 resonance) include horseshoe orbits and Trojan orbits. These are quite interesting but we will not discuss them further because of time limitations.

The next strongest resonances occur when \( H_{kmn} \sim e \) and require \( m = k \pm 1 \). For these \( H_1 \) can be approximated as

\[ H_1 = \sum_m \sqrt{L - GA_m} \cos[(m \mp 1)\ell + m\omega + m\Omega - mn_*t] \\
= \sum_m \sqrt{L - GA_m} \cos[(m \mp 1)\lambda \pm \varpi - mn_*t] \] (22)

where \( A_m \) is a constant (in principle \( A_m \) depends on \( L \) but since the variations in \( L \) are small we can treat \( A_m \) as a constant). Resonance occurs when \((m \mp 1)n = mn_c\). There are also resonances with \((m \mp j)n_c = mn_c\) with \( j > 1 \), but these are weaker since the resonant Hamiltonian is \( \sim e^j \) by the d’Alembert property.

Since we have assumed that the test particle is interior to the planet, and since we can restrict ourselves without loss of generality to \( m \geq 0 \), we must choose the upper sign, so \( n = mn_c/(m-1) \) or \( n/n_c = 2, \frac{3}{2}, \frac{4}{3}, \ldots \). These are known as mean-motion or Lindblad resonances.

We can simplify the problem by a canonical transformation using the generating function

\[ S = -J_\star[(m - 1)\ell + m\omega + m\Omega - mn_*t] + J_{f1}(\ell + \omega) + J_{f2}(\ell + \omega + \Omega). \] (23)
The new coordinates are
\[
\theta_s = \frac{\partial S}{\partial J_s} = -(m - 1)\ell - m\omega - m\Omega + mn_c t, \quad \theta_{f1} = \ell + \omega, \quad \theta_{f2} = \ell + \omega + \Omega. \tag{24}
\]
The subscript “s” indicates that \(\theta_s\) varies slowly near the resonance while “f” indicates that \(\theta_{f1}\) and \(\theta_{f2}\) vary rapidly (\(\dot{\theta}_{f1} \approx \dot{\theta}_{f2} \approx n\)). The new momenta are related to the Delaunay momenta by
\[
L = \frac{\partial S}{\partial \ell} = (1 - m)J_s + J_{f1} + J_{f2}, \quad G = \frac{\partial S}{\partial \omega} = -mJ_s + J_{f1} + J_{f2}, \quad H = -mJ_s + J_{f2}. \tag{25}
\]
Solving for the new momenta,
\[
J_s = L - G, \quad J_{f1} = G - H, \quad J_{f2} = H + m(L - G), \tag{26}
\]
and the new Hamiltonian is
\[
H = H_K + H_1 + \frac{\partial S}{\partial t} = -\frac{(GM)^2}{2[(1 - m)J_s + J_{f1} + J_{f2}]^2} + A_m\sqrt{J_s \cos \theta_s + mn_c J_s}. \tag{27}
\]
The Hamiltonian is independent of the fast coordinates \(\theta_{f1}\) and \(\theta_{f2}\) so \(J_{f1}\) and \(J_{f2}\) are constants. Since the inclination is zero \(G = H\); thus \(J_{f1} = 0\). Moreover since the eccentricity is small \(J_s\) is small and \(J_{f2} \simeq L\), which is not small. Thus we can expand the denominator in equation (27) to quadratic order in \(J_s/J_{f2}\). Dropping an unimportant constant term, we have
\[
H = J_s[mn_c - (m - 1)n_r] - \frac{3(m - 1)^2}{2a_r^2}J_s^2 + A_m\sqrt{J_s \cos \theta_s}; \tag{28}
\]
here we have defined the constants \(a_r\) and \(n_r\) by \(J_{f2} = (GMa_r)^{1/2}\) and \(n_r = (GM/a_r^3)^{1/2}\). When the perturbing potential \(A_m\) is small and the eccentricity is also small (so \(J_s\) is small), we have \(\dot{\theta}_s \approx \partial H/\partial J_s = mn_c - (m - 1)n_r\) so this constant can be thought of as the distance away from exact resonance, where \(\dot{\theta}_s = 0\).

\[
H = -3\Delta R + R^2 - 2\sqrt{2R}\cos r, \tag{29}
\]
where \(R\) and \(r\) is a conjugate momentum-coordinate pair and \(\Delta \propto mn_c - (m - 1)n_r\) measures the distance away from resonance. Henrard & Lemaître call this the “second fundamental model for resonance” in contrast to the “first fundamental model for resonance” which is the simple pendulum,
\[
H = -\Delta R + \frac{1}{2}R^2 - \cos r. \tag{30}
\]
The Hamiltonian (29) has only one degree of freedom and thus is integrable. Thorough discussions of its solutions and how they evolve under slow changes in $\Delta$ (as might be caused, e.g., by migration or growth of the mass of the planet) are given in the references above. They provide an analytical description of such phenomena as resonance capture, eccentricity excitation during resonance crossing, etc. I shall not discuss these results due to lack of time, and since they are covered in the references. The main purpose of this discussion has been to show how perturbation theory can be used to derive a model Hamiltonian that is simple enough to solve analytically but realistic enough to capture the main features of mean-motion resonances.