**DNA, History, and Archaeology**

**BY NICOLA DI COSMO**

**Hist**orians today can hardly answer the question: when does history begin? Traditional boundaries between history, prehistory, and protohistory have been blurred if not completely erased by the rise of concepts such as “Big History” and “macrohistory.” If even the Big Bang is history, connected to human evolution and social development through a chain of geological, biological, and ecological events, then the realm of history, while remaining firmly anthropocentric, becomes all-embracing.

An expanding historical horizon that, from antiquity to recent times, attempts to include places far beyond the sights of literate civilizations and traditional caesuras between a history illuminated by written sources and a prehistory of stone, copper, and pots has forced history and prehistory to coexist in a rather inelgant embrace. Such a blurring of the boundaries between those human pasts that left us more or less vivid and abundant written records, and other pasts, which, on the contrary, are knowable only through the spadework and fieldwork of enterprising archaeologists, ethnographers, and anthropologists, has also changed (or is at least threatening to change) the nature of the work of professional historians.

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**Knots and Quantum Theory**

**BY EDWARD WITTEN**

In everyday life, a string—such as a shoelace—is usually used to secure something or hold it in place. When we tie a knot, the purpose is to help the string do its job. All too often, we run into a complicated and tangled mess of string, but ordinarily this happens by mistake.

The term “knot” as it is used by mathematicians is abstracted from this experience just a little bit. A knot in the mathematical sense is a possibly tangled loop, freely floating in ordinary space. Thus, mathematicians study the tangle itself. A typical knot in the mathematical sense is shown in Figure 1. Hopefully, this picture reminds us of something we know from everyday life. It can be quite hard to make sense of a tangled piece of string—to decide whether it can be untangled and if so how. It is equally hard to decide if two tangles are equivalent.

Such questions might not sound like mathematics, if one is accustomed to thinking that mathematics is about adding, subtracting, multiplying, and dividing. But actually, in the twentieth century, mathematicians developed a rather deep theory of knots, with surprising ways to answer questions like whether a given tangle can be untangled.

(Continued on page 4)

**“Spontaneous Revolution” in Tunisia**

**YEARNINGS FOR FREEDOM, JUSTICE, AND DIGNITY**

**BY MOHAMED NACHTI**

The Tunisian revolution of 2011 (al-thawra al-tunisiya) was the result of a series of protests and insurrectional demonstrations, which started in December 2010 and reached culmination on January 14, 2011, with the flight of Zine el-Abidine Ben Ali, the dictator who had held power for twenty-three years. It did not occur in a manner comparable to other revolutions. The army, for instance, did not intervene, nor were there actions of an organized rebellious faction. The demonstrations were peaceful, although the police used live ammunition, bringing the death toll to more than one hundred.

The demonstrations began in the town of Sidi Bouzid, west of the country’s geographical center. On December 17, 2010, a young street vendor set himself on fire following the confiscation of his wares (fruits and vegetables) by the police. Mohamed Bouazizi was twenty-six, and he succumbed to his burns on January 4. The next day, five thousand people attended his funeral. He became the symbol of the liberation of the Tunisian people from the despotic rule of the Ben Ali regime. The population, and predominantly the youth, began to demonstrate with calm determination, in order to force the dictator who had ruled for twenty-three years, fled on January 14, 2011.

Protests in Tunisia culminated when Zine el-Abidine Ben Ali, who had ruled for twenty-three years, fled on January 14, 2011.

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KNOT THEORY (Continued from page 1)

one could make such a claim. For example, no one would try to explain Andrew Wiles’s proof of Fermat’s Last Theorem to high school students.

To simplify slightly (see the box on page 5 for more details), what Jones discovered was a way to calculate a number for every knot. Let us call our knot \( K \), and we will write \( J_K \) for the number that Jones calculates for this knot. There is a definite rule that allows you to calculate \( J_K \) for any knot. No matter how complicated \( K \) may be, one can calculate \( J_K \) if one is patient enough.

If \( J_K \) is not equal to 1, then the knot \( K \) can never be untangled. For example, let us go back to the knot that was sketched in Figure 1. If you were to think about how to untie that particular knot, you certainly would not succeed. But how could one prove that it is impossible? Jones gave a way to answer this sort of question: calculate \( J_K \), and if it is not equal to 1, then the knot \( K \) can never be untied. Jones’s method of computing \( J_K \) was very clever, but once it was found, anyone could use it without any particular cleverness, just by following instructions.

In fact, there are an astonishing variety of ways to calculate \( J_K \). I will explain just one of the simplest. One important rule applies to the “unknot,” which is a simple untangled loop (Figure 2). If \( K \) is an unknot then \( J_K = 1 \).

For all other knots, we have to play a little game. To start, we pick three favorite numbers, for example 2, 3, and 10. Now we are going to do something that might seem to make life more complicated. Instead of a single knot \( K \), we are going to consider three knots \( K, K’, \) and \( K” \). If the three knots that we pick are related in a certain way, there will be an arithmetic relationship

\[
2J_K + 3J_{K'} + 10J_{K”} = 0
\]

This relationship—or, as mathematicians call it, this identity—is so powerful that it enables us to calculate the \( J’s \).

How should \( K, K’, \) and \( K” \) be related in order to participate in such an identity? In Figure 3, I have drawn a partial knot. It is only a partial knot because there is a missing piece, indicated by the question mark. There are many possible ways to complete the knot by filling in the missing piece. In Figure 4, I have sketched three of the simplest ways to do this. Choosing one of these three fillings gives us a knot that we call \( K, K’, \) or \( K” \), respectively, and then, as stated previously, we declare that Jones’s numbers \( J_K, J_{K’}, \) and \( J_{K”} \) obey the relationship \( 2J_K + 3J_{K’} + 10J_{K”} = 0 \).

It turns out that this is a rather powerful relationship, which enables one to calculate \( J_K \) for any \( K \). The details of this are explained somewhat more fully in the box (on page 5).

The surprise here is not so much that this rule can be used to calculate \( J_K \), but that in doing this one never runs into a contradiction. One could anticipate a contradiction because actually there are many ways to use the properties that I have described to calculate \( J_K \). However, Jones and other mathematicians showed in the 1980s that there is never a contradiction—one always arrives at the same answer for \( J_K \) no matter how one uses the procedure just described (or the other, related, procedures that were discovered in that period) to calculate it.

These proofs showed that the recipe for calculating \( J_K \) were correct, but they left a “why” question. Unfortunately, it is not that easy to explain to someone who does not work in mathematics or physics that the field the difference between knowing “what” is true and knowing “why” it is true. Yet the beauty of the “why” answers is much of the reason that people do mathematics.

In this case, as people worked on the Jones polynomial, they discovered more and more remarkable formulas, with less and less clarity about what they meant. But there was a clue. In fact, there were a lot of clues. As the subject developed, beginning with Jones’s original work, it had many ties with mathematical physics. Sometimes and one time dimension.

Figure 2

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An important point to emphasize is that we are relativistic physicists, since relativity was also invented in the twentieth century, along with quantum mechanics. So when I draw a path, it is really a path in spacetime, not a path in space.

The physical dimension of the real world we live in is therefore three—three space dimensions and one dimension of time. But to understand knot theory, at least for the moment, we are going to imagine a world of only three spacetime dimensions—two space dimensions and one time dimension.

In a world of three spacetime dimensions, the particle path might be knotted. For an example of a knotted path, see Figure 6.

A quantum physicist has to sum the effects of all possible paths by which a particle might reach its destination. How to calculate such a sum is what physicists learned in constructing quantum theory and what is now the Standard Model of particle physics.

Quantum mechanically, though any path is possible, if the particle traveled on a particular path \( K \), then there is a “probability amplitude” for it to arrive at its destination, and this amplitude depends on \( K \). The way that the amplitude depends on \( K \) is very important—it is the reason that there is some order even in a quantum universe. All paths are possible, but peculiar ones with a lot of zigzags are not very likely.

The quantum mechanical amplitude that the particle traveled on a path \( K \) is given by something called the Wilson operator, \( W_K \). For our purposes, we really do not need to know how it is defined. All we need to know is that it is a basic ingredient in quantum physics; for instance, physicists use it in calculating the force between quarks.

The connection between the Jones polynomial and quantum physics turns out to be simply that if we regard a knot \( K \) as the orbit in spacetime of a charged particle, then the Jones polynomial is the average value of the Wilson operator. Thus the quantum formula for the Jones polynomial is just \( \langle W_K \rangle = <W_K> \), where the symbol \( < > \) represents a process of quantum averaging.

When one carries out this program, the version of quantum theory that is relevant uses something called the Chern-Simons function for gauge fields. (Both Shing-Tien Chan, who founded much of modern differential geometry, and James Simons, now an Institute Trustee, are former IAS Members.) This story as I have told it so far goes back to my early years at the Institute. But there is actually a more contemporary twist to this tale. This is the reason that it seems timely to write about this topic now.

In everyday life, a knot is a physical object that exists in space, but to interpret the Jones polynomial in terms of quantum theory, we have instead had to view a knot as a path in spacetime, one of only three dimensions. This is perhaps a less obvious viewpoint about what a knot means.

However, around 1990, when he was a Member at the Institute, Igor Frenkel started to develop what he hoped would be a new mathematical theory in which the knot would indeed be seen as a physical object rather than a path in spacetime. The new theory was supposed to involve a more powerful version of the Jones polynomial.

I wish I could say that I gave Frenkel some useful advice, but all I did was to tell him that it would not work because the Chern-Simons function is special to three dimensions and does not have the right sort of extension to four dimensions. It actually was a sensible objection at the time, and I am still surprised that it turned out to be wrong.
Anyway, Frenkel continued to develop this idea with, among others, his student Mikhail Khovanov and IAS Member Louis Crane. Finally, around the year 2000, Khovanov created what is now known as Khovanov homology—a refinement of the Jones polynomial in which the knot is a physical object in four spacetime dimensions rather than the path of a point particle in a spacetime of only three dimensions.

Khovanov homology is like the Jones polynomial in that once it was invented, it could be computed by an explicit set of rules, though these rules are far more sophisticated than the ones that go into the Jones polynomial. I do not think that one would try to

even today.

Edward Witten, Charles Simonyi Professor in the School of Natural Sciences, first came to the Institute as a Member in 1984 and was appointed as a Professor in 1987. His work has significantly enriched the fields of mathematics and physics, and he has contributed greatly to the modern interest in superstrings as a candidate theory for the unification of all known physical interactions. Most recently, he has explored quantum duality symmetries of field theories and string theories, opening significant new perspectives on particle physics, string theory, and topology. A video of a talk that Witten gave to the Friends of the Institute for Advanced Study on quantum theory and knots is available at http://video.ias.edu/witten-friends.

explain the definition of Khovanov homology to high school students.

Knot theory has had a great deal of impact mathematically. For example, it was a major topic in a special program at the School of Math at the Institute a few years ago.

One does not need quantum physics to define Khovanov homology, though one may need quantum physics to understand what it means. Indeed, in 2003, the physicists Sergei Gukov, Albert Schwarz, and Cumrun Vafa proposed a quantum interpretation of Khovanov homology, based on the earlier work of Vafa with Hirosi Ooguri. (These physicists are all former IAS Members and/or students.) Their story used plenty of avant-garde ideas about quantum fields and strings and all that.

As beautiful and powerful as their story is, I've always suspected that there might be a more direct route, and I spent the last year trying to construct one. Though in a sense I succeeded, I am not sure whether to say that I found a more direct route or just a somewhat different one.1

The main difference between Khovanov homology and the Jones polynomial is that the goal of Khovanov homology is more abstract. While the Jones polynomial of a knot K is a number JK, the Khovanov homology of K is a "space of quantum states" known as HK. If you think of a knot as a physical object in three-dimensional space, then HK is the space of its possible quantum states.

Because Khovanov homology is in four spacetime dimensions rather than three, it involves ideas that are even closer to real particle physics than those that go into understanding the Jones polynomial. One important idea is symmetry between electric and magnetic fields. This is called electric-magnetic duality, and was pioneered in the 1970s by Peter Goddard (current Director of the Institute), Jean Nuys, and David Olive (all former IAS Members). Since the mid-1990s, it has been one of the main tools in studies of quantum fields and strings, at the Institute and elsewhere. The use of electric-magnetic duality is actually crucial in circumventing the obstacle that had convinced me twenty years ago that Igor Frenkel's idea could not work.

Another facet of string theory also turns out to be important: extra dimensions. Even though what we want is supposed to be a theory in four spacetime dimensions, it turns out that understanding it properly involves relating it to theories in five or six dimensions.

The biggest surprise of all is that even though it can be defined by an explicit recipe with no reference to quantum physics—and that is how it was discovered—Khovanov homology can be understood, possibly much better, using the most modern tools of quantum field theory and string theory. Probably the full story involves physics ideas that we still do not entirely understand even today.


Some More Mathematical Details

To define the Jones polynomial (or actually a generalization of it called the HOMFLYPT polynomial) using the sort of relation described in the text, we really should introduce three variables a, b, and c and consider a general identity a JK + b JK' + c JK" = 0. By virtue of this identity (and the normalization that JK = 1 for an unknot), JK turns out to be a homogeneous, rational function of the three variables.

JK is defined for a link (a disjoint union of embedded circles in three-dimensional space) and not only for a knot (a single embedded circle). In general, some or all of K, K', and K" may have more than one component. In the original Jones polynomial, the knots and links are unoriented, as shown in the illustrations that accompany this article, but in the general HOMFLYPT polynomial they are oriented.

The identity a JK + b JK' + c JK" = 0 (for every triple K, K', K" related as in Figure 4) suffices to determine the Jones polynomial of any link. This can be proved by induction in the number of crossings when the link is projected to two dimensions. For example, a knot with three crossings is sketched in Figure 7. In the special case b = 0, the relation a JK + c JK" = 0 says that any two strands can be passed through each other while multiplying the Jones polynomial by −c/a. Of course, if strands can freely be passed through each other, we cannot use any knot. For b nonzero, the relationship has an extra term, but this extra term reduces the number of crossings. So inductively, one reduces to the case that K is a link with no crossings at all—in other words, a union of s disjoint circles in the plane, for some s. A simple application of the identity a JK + b JK' + c JK" = 0 shows that in this special case, JK = ((−a+bc)/b)s).

The proof that the identity a JK + b JK' + c JK" = 0 leads to a consistent result no matter how one applies it is less obvious, and is made by showing that the identity is consistent with certain relations among links that are known as the Reidemeister moves.

—Edward Witten

The Institute Woods: Photographs by Vladimir Voevodsky

An exhibit now on view at the Institute features photographs of the Institute Woods taken by Vladimir Voevodsky, Professor in the School of Mathematics, illustrating the remarkable diversity of its insect, bird, animal, and plant life. For information about the Woods, visit www.ias.edu/about/institute-grounds.