Anomalies, Conformal Manifolds, and Spheres

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Jaume Gomis, Po-Shen Hsin, Zohar Komargodski, Adam Schwimmer, NS, Stefan Theisen, arXiv:1509.08511
CFT Sphere partition function $\log Z$

- Power divergent terms are not universal. Can be removed by counter terms like $\Lambda^d \int \sqrt{g} + \Lambda^{d-2} \int \sqrt{g} R + \ldots$

- In addition
  - for odd $d$ $\log Z = -F$ is universal (ambiguity in quantized imaginary part due to Chern-Simons terms – depends on framing)
  - for even $d$ $\log Z = C \log(r\Lambda) - F$.

  - $C$ is universal (in $2d$ it is $c/3$ and in $4d$ it is $-\alpha$)
  - $F$ is not universal; it can be absorbed in a local counter-term, $\int \sqrt{g} E_d F$ with $E_d$ the Euler density

- Used in $c$-theorem and its generalizations, entanglement entropy, ...
Conformal manifolds

- $S = S_0 + \int \lambda^i O_i(x)$ is an exactly marginal deformation
- Family of CFTs labeled by coordinates $\lambda^i$
- Metric on the conformal manifold – the Zamolodchikov metric

$$\langle O_i(x) O_j(0) \rangle = \frac{g_{ij}(\lambda)}{|x|^{2d}}$$

- Focus on $d = 2$
Conformal manifolds of $2d$ $(2,2)$ SCFT

- Typical example: sigma models with Calabi-Yau target space.
- In string theory the coordinates on the conformal manifold are the moduli – massless fields in $4d$
  - $2d$ chiral fields $\lambda$
  - $2d$ twisted chiral fields $\tilde{\lambda}$
- The conformal manifold is Kahler with

$$K = K_c(\lambda, \bar{\lambda}) + K_{tc}(\tilde{\lambda}, \bar{\tilde{\lambda}})$$
2d (2,2) curved superspace

• For rigid SUSY in curved spacetime use supergravity [Festuccia, NS]

• Simplification in 2d
  – (locally) pick conformal gauge $g_{\mu\nu} = e^{2\sigma} \delta_{\mu\nu}$.
  – for SUSY (locally) pick superconformal gauge.

• Two kinds of (2,2) supergravities. In the superconformal gauge they depend on
  – A chiral $\Sigma = \sigma + i \, a + \cdots$ with $A_\mu = \epsilon_{\mu\nu} \partial^\nu a$ an axial R-gauge field (Lorentz gauge). We will focus on this.
  – A twisted chiral $\widetilde{\Sigma} = \sigma + i \, \tilde{a} + \cdots$ with $\widetilde{A}_\mu = \epsilon_{\mu\nu} \partial^\nu \tilde{a}$ a vector R-gauge field (Lorentz gauge).
2d (2,2) curved superspace

• In this language the curvature is a chiral superfield $\mathcal{R} = \bar{D}^2 \bar{\Sigma}$

• To preserve rigid SUSY in a non-conformal theory we need to add terms to the flat superspace Lagrangian

• Various backgrounds in the literature are easily described, e.g.
  – Topological twist is $\Sigma = 0$, $\bar{\Sigma} \neq 0$
  – Omega background
  – Supersymmetry on $\mathbb{S}^2$ ([Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee]) is achieved with (suppress $r$ dependence)
    $$\Sigma = - \log(1 + |z|^2) + \theta^2 \frac{i}{1 + |z|^2}$$
    $$\bar{\Sigma} = - \log(1 + |z|^2) + \bar{\theta}^2 \frac{i}{1 + |z|^2}$$
  – $\bar{\Sigma}$ is not necessarily the complex conjugate of $\Sigma$
(2,2) sphere partition functions

- Amazing conjecture [Jockers, Kumar, Lapan, Morrison, Romo]: the $S^2$ partition function with the background of [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee] is

\[ Z = r^{c/3} e^{-K_c(\lambda, \bar{\lambda})} \]

(restoring the radius $r$, whose power reflects the ordinary conformal anomaly).

Similarly, using $\tilde{\Sigma}$ it is

\[ Z = r^{c/3} e^{-K_{tc}(\tilde{\lambda}, \tilde{\bar{\lambda}})}. \]

- Proofs based on localization, squashed sphere, $tt^*$, twisting, counter terms and properties of the background [Gomis, Lee; Gerchkovitz, Gomis, Komargodski; ...]
Questions/confusions

• Given that the one point function of a marginal operator vanishes, how can the sphere partition function depend on $\lambda$?

• Why is it meaningful?
  – Can add a local counter term $\int \sqrt{g} R f(\lambda, \bar{\lambda})$, making the answer non-universal

• In an SCFT on the sphere there is no need to add terms to the Lagrangian to preserve SUSY
  – Why does it depend on the background $\Sigma$?
  – If it does not, what determines whether we used $\Sigma$ or $\tilde{\Sigma}$ to find $e^{-K_c}$ or $e^{-K_{tc}}$?
  – Where is the freedom in Kahler transformations?

• What’s the conceptual reason for it? Is it UV or IR?
Conformal manifolds and anomalies (w/o SUSY)

Zamolodchikov metric

\[ \langle O_i(x) \ O_j(0) \rangle = \frac{g_{ij}(\lambda)}{|x|^4} \]

In momentum space

\[ \int e^{ipx} \langle O_i(x) \ O_j(0) \rangle \sim g_{ij}(\lambda)p^2 \log(\mu^2/p^2) \]

Dependence on the scale \( \mu \) leads to a conformal anomaly: with position dependent \( \lambda \) the variation with respect to the conformal factor \( \delta \sigma \) is (suppressed coefficients) [Osborn; Friedan, Konechny]

\[ \delta S_{\text{eff}} \sim \int \delta \sigma \left( c \ R + g_{ij}(\lambda) \partial_\mu \lambda^i \partial^\mu \lambda^j + \ldots \right) \]

Ordinary conformal anomaly  A more subtle anomaly
(2,2) SCFT

- The anomaly should be expressed in superspace – need to use curved superspace.
- Focus on the supergravity that in the superconformal gauge uses a chiral superfield $\Sigma$
- The supersymmetrization of the anomaly is

$$\delta S_{eff} \sim \int d^2x \, d^4\theta \, (\delta \Sigma + \delta \bar{\Sigma}) \left( c \, (\Sigma + \bar{\Sigma}) + K_c(\lambda, \bar{\lambda}) - K_{tc}(\bar{\lambda}, \bar{\lambda}) \right)$$

- It is invariant under Kahler transformations of $K_{tc}(\bar{\lambda}, \bar{\lambda})$, but not under Kahler transformations of $K_c(\lambda, \bar{\lambda})$. 
Ambiguities

- In the superconformal gauge the local terms are expressed in terms of the chiral curvature superfield $\mathcal{R} = \bar{D}^2 \bar{\Sigma}$. 
- Terms that depend on $\Sigma$ not through $\mathcal{R}$ are non-local. 
- Therefore, the anomaly is not a variation of a local term. 
- Freedom in the local term

$$\int d^2 \theta \mathcal{R} f(\lambda) + c.c. = \int d^4 \theta \bar{\Sigma} f(\lambda) + c.c.$$ 

leads to freedom in Kahler transformations of $K_c(\lambda, \bar{\lambda})$. Other than that, the anomaly is unambiguous.
The anomaly in components

For a purely conformal variation $\delta \Sigma = \delta \sigma$ the anomaly is

$$\delta S_{eff} \sim \int d^4 \theta (\delta \Sigma + \delta \bar{\Sigma}) \left( c (\Sigma + \bar{\Sigma}) + K_c (\lambda, \bar{\lambda}) - K_{tc} (\bar{\lambda}, \bar{\lambda}) \right)$$

$$= \int \left[ \delta \sigma \left( c \Box \sigma + g_{ii} \partial_\mu \lambda^i \partial^\mu \bar{\lambda}^i + \tilde{g}_{\bar{a}\bar{a}} \partial_\mu \bar{\lambda}^\bar{a} \partial^\mu \bar{\lambda}^{\bar{a}} \right) \right.$$

$$- \Box \delta \sigma K_c (\lambda, \bar{\lambda}) \right]$$

The last term leads to

$$S_{eff} \sim \int R K_c (\lambda, \bar{\lambda}) + \ldots$$

and hence on $\mathbb{S}^2$

$$Z = r^{c/3} e^{-K_c}$$

Q.E.D.
Extensions

- Trivial to repeat with \( \tilde{\Sigma} \) and to find \( K_{tc} \)
- Can reproduce that for \( 4d \mathcal{N} = 2 \) [Gerchkovitz, Gomis, Komargodski; Gomis Ishtiaque]

\[
Z = r^{-a} e^{K(\tau,\bar{\tau})/12}
\]

Ordinary conformal anomaly \quad The more subtle anomaly

- \( 2d \mathcal{N} = (0,2) \)
Conclusions

• Anomaly under conformal transformations when the coupling constants depend on position
  – Unrelated to supersymmetry
  – This is a UV phenomenon
    • Visible on flat $\mathbb{R}^2$
    • Independent of the background
• Supersymmetry restricts
  – the form of the anomaly
  – the ambiguity due to local counter terms
Conclusions

• The $S^2$ partition function depends on the anomaly.
  – New derivation of
    
    $Z = r^{c/3} e^{-Kc}$
  
    • The usual conformal anomaly
    • The more subtle conformal anomaly
  – Addresses the questions/confusions we raised

• Three step process
  – the anomaly
  – the ambiguity (freedom in counter terms)
  – the sphere

• Several extensions