Anomalies, Conformal Manifolds, and Spheres

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Jaume Gomis, Po-Shen Hsin, Zohar Komargodski, Adam Schwimmer, NS, Stefan Theisen, arXiv:1509.08511
CFT Sphere partition function $\log Z$

• Power divergent terms are not universal. Can be removed by counterterms like $\Lambda^d \int \sqrt{\gamma} + \Lambda^{d-2} \int \sqrt{\gamma} R + \cdots$

• In addition
  – for odd $d$ $\log Z = -F$ is universal (ambiguity in quantized imaginary part due to Chern-Simons terms – depends on framing)
  – for even $d$ $\log Z = C \log(r \Lambda) - F$.

  • $C$ is universal (in $2d$ it is $c/3$ and in $4d$ it is $-a$)
  • $F$ is not universal; it can be absorbed in a local counterterm, $\int \sqrt{\gamma} E_d F$ with $E_d$ the Euler density

• Used in c-theorem and its generalizations, entanglement entropy, ...
Conformal manifolds

• $S = S_0 + \int \lambda^i O_i(x)$ is an exactly marginal deformation

• Family of CFTs labeled by coordinates $\lambda^i$

• Metric on the conformal manifold – the Zamolodchikov metric

$$\langle O_i(x) O_j(0) \rangle = \frac{g_{ij}(\lambda)}{|x|^{2d}}$$

• Focus on $d = 2$ (later $d = 4$)
Conformal manifolds of 2d (2,2) SCFT

• Typical example: sigma models with Calabi-Yau target space.
• In string theory the coordinates on the conformal manifold are the moduli – massless fields in 4d
  – 2d chiral fields $\lambda$
  – 2d twisted chiral fields $\tilde{\lambda}$
• The conformal manifold is Kahler with

$$K = K_c(\lambda, \bar{\lambda}) + K_{tc}(\tilde{\lambda}, \bar{\tilde{\lambda}})$$
2d (2,2) curved superspace

- For rigid SUSY in curved spacetime use supergravity [Festuccia, NS]
- Simplification in 2d
  - (locally) pick conformal gauge $\gamma_{\mu\nu} = e^{2\sigma} \delta_{\mu\nu}$
  - for SUSY (locally) pick superconformal gauge
  - use flat space expressions with explicit $\sigma$'s
- Two kinds of (2,2) supergravities. In the superconformal gauge they depend on
  - A chiral $\Sigma = \sigma + i a + \cdots$ with $A_\mu = \epsilon_{\mu\nu} \partial^\nu a$ an axial R-gauge field (Lorentz gauge). The curvature is in a chiral superfield $\mathcal{R} = \overline{D}^2 \bar{\Sigma}$. We will focus on this.
  - A twisted chiral $\tilde{\Sigma} = \sigma + i \tilde{a} + \cdots$ with $\tilde{A}_\mu = \epsilon_{\mu\nu} \partial^\nu \tilde{a}$ a vector R-gauge field (Lorentz gauge).
2d (2,2) curved superspace

• In this language the curvature is in a chiral superfield $\mathcal{R} = \overline{D}^2 \Sigma$. It contains $\partial^2 \sigma$.

• To preserve rigid SUSY in a non-conformal theory we need to add terms to the flat superspace Lagrangian.

• Various backgrounds in the literature are easily described, e.g.
  – Topological twist is $\Sigma = 0$, $\bar{\Sigma} = 2\sigma \neq 0$
  – Omega background

\[ \bar{\Sigma} = 2\sigma + 2i\epsilon \bar{z} \partial_{\bar{z}} (2\sigma + \log \bar{z}) \bar{\theta}^2 \]
2d (2,2) curved superspace

– Supersymmetry on $S^2$ [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee] is achieved with (suppress $r$ dependence)

\[
\Sigma = - \log(1 + |z|^2) + \theta^2 \frac{i}{1+|z|^2}
\]

\[
\bar{\Sigma} = - \log(1 + |z|^2) + \bar{\theta}^2 \frac{i}{1+|z|^2}
\]

– $\bar{\Sigma}$ is not necessarily the complex conjugate of $\Sigma$. 
(2,2) sphere partition functions

• [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee] placed a nonconformal gauged linear sigma model on a sphere with

\[
\Sigma = -\log(1 + |z|^2) + \theta^2 \frac{i}{1+|z|^2}
\]

\[
\bar{\Sigma} = -\log(1 + |z|^2) + \bar{\theta}^2 \frac{i}{1+|z|^2}
\]

• This theory flows in the IR to a nonlinear sigma model with Calabi-Yau target space.

• They computed \( Z \) of the IR theory using localization in the UV theory.
(2,2) sphere partition functions

- Amazing conjecture [Jockers, Kumar, Lapan, Morrison, Romo]: this $S^2$ partition function is

\[ Z = r^{c/3} e^{-K_c(\lambda,\bar{\lambda})} \]

(restoring the radius $r$, whose power reflects the ordinary conformal anomaly).

Similarly, using $\tilde{\Sigma}$ it is $Z = r^{c/3} e^{-K_{tc}(\tilde{\lambda},\bar{\tilde{\lambda}})}$.

- Proofs based on localization, squashed sphere, $tt^*$, twisting, counterterms and properties of the background [Gomis, Lee; Gerchkovitz, Gomis, Komargodski; ...].
Questions/confusions

• Given that the one point function of a marginal operator vanishes, how can the sphere partition function depend on $\lambda$?
• Why is it meaningful?
  – Can add a local counterterm $\int \sqrt{\gamma} \ R \ f(\lambda, \bar{\lambda})$, making the answer non-universal
• In an SCFT on the sphere there is no need to add terms to the Lagrangian to preserve SUSY
  – Why does it depend on the background $\Sigma$?
  – If it does not, what determines whether we used $\Sigma$ or $\tilde{\Sigma}$ to find $e^{-Kc}$ or $e^{-Ktc}$?
  – Where is the freedom in Kahler transformations?
• What’s the conceptual reason for it? Is it UV or IR?
Conformal manifolds and anomalies (w/o SUSY)

Zamolodchikov metric \( \langle O_i(x) \, O_j(0) \rangle = \frac{g_{ij}(\lambda)}{|x|^4} \)

In momentum space \( \int e^{ipx} \langle O_i(x) \, O_j(0) \rangle \sim g_{ij}(\lambda)p^2 \log(\mu^2/p^2) \)

Dependence on the scale \( \mu \) leads to a conformal anomaly: with position dependent \( \lambda \) (suppressed coefficients) [Osborn; Friedan, Konechny]

\[ T_\mu^\mu = c \, R + g_{ij}(\lambda) \partial_\mu \lambda^i \, \partial^\mu \lambda^j + \ldots \]

Ordinary conformal anomaly

A more subtle anomaly (actually, less subtle)
More about anomalies

• The partition function $Z$ is a nonlocal functional of the background fields (the metric $\gamma_{\mu\nu}$, exactly marginal couplings $\lambda^i$, background gauge fields, etc.).

• Its variation under changing the conformal factor $\delta_\sigma \log Z$ is an integral of a local functional of the background fields and $\delta \sigma$.

• $\delta_\sigma \log Z$
  – Must be coordinate invariant in spacetime (assume that the regularization preserves it, i.e. $T_{\mu\nu}$ is conserved also at coincident points)
  – Must be coordinate invariant in the conformal manifold
  – Must obey the Wess-Zumino consistency conditions
  – In SUSY theories it must be supersymmetric
More about anomalies

• A term in $\delta_\sigma \log Z$ that is a Weyl variation of a local term is considered trivial.

• An anomaly is a “cohomologically nontrivial” term.
  – It cannot be removed by changing a local counterterm.
  – It cannot change by changing the renormalization scheme.

• Therefore, even though it arises due to a short distance regulator, it is universal – it does not depend on the choice of regulator.
Returning to conformal manifolds and anomalies (w/o SUSY)

\[
\langle O_i(x) \, O_j(0) \rangle = \frac{g_{ij}(\lambda)}{|x|^4}
\]

\[
\int e^{ipx} \langle O_i(x) \, O_j(0) \rangle \sim g_{ij}(\lambda)p^2 \log(\mu^2/p^2)
\]

leads to [Osborn; Friedan, Konechny]

\[
\delta_\sigma \log Z \sim \int \sqrt{g} \, \delta \sigma \left( c \, R + g_{ij}(\lambda) \, \partial_\mu \lambda^i \, \partial^\mu \lambda^j + \ldots \right)
\]

Ordinary conformal anomaly  A more subtle anomaly
Conformal manifolds and anomalies (w/o SUSY)

Another possible anomaly in 2d
$$\delta_\sigma \log Z \sim \int \sqrt{\gamma} \delta \sigma \epsilon^{\mu\nu} B_{ij}(\lambda) \partial_\mu \lambda^i \partial_\nu \lambda^j$$
can be ruled out as follows.

$$\langle O_{i_1} (p_1) O_{i_2} (p_2) \cdots \rangle = \log \mu^2 \delta (\sum p_i) A_{i_1 i_2} \cdots + \cdots$$

where $\cdots$ are finite.

$A_{i_1 i_2} \cdots$ is a second order polynomial in the momenta.

Set e.g. $p_3 = p_4 = \cdots = 0$ and find

$$A_{i_1 i_2} \cdots = p_1^2 \partial_{i_3} \partial_{i_4} \cdots g_{i_1 i_2}.$$ 

Therefore, all these anomalies are generated from the metric and there is no “$B$-field” anomaly.
Warmup: $2d \mathcal{N} = 1$

- The conformal manifold is parameterized by real superfields $\lambda^i$.
- The anomaly should be expressed in superspace – need to use curved superspace.
- Use the superconformal gauge – the conformal factor $\sigma$ is in a real superfield $\Sigma$ and the curvature is in $\mathcal{R} = D^2 \Sigma$.
- The anomaly $\delta_\sigma \log Z \sim \int \sqrt{\gamma} \, \delta \sigma \left( c \, R + g_{ij}(\lambda) \, \partial_\mu \lambda^i \, \partial^\mu \lambda^j + \cdots \right)$ is supersymmetrized in the superconformal gauge as $\delta_\Sigma \log Z \sim \int d^2 \theta \, \delta \Sigma \left( c \, D^2 \Sigma + g_{ij}(\lambda) \, D_+ \lambda^i \, D_- \lambda^j + \cdots \right)$

Ordinary conformal anomaly

A more subtle anomaly

- Ambiguity due to a local counterterm $\int d^2 \theta \, \mathcal{R} \, f(\lambda)$ prevents us from making any statement about $Z$. 
The conformal manifold is parameterized by chiral superfields $\lambda^i$ and twisted chiral superfields $\tilde{\lambda}^{\bar{a}}$.

The anomaly should be expressed in superspace – need to use curved superspace.

Focus on the supergravity, where the conformal factor $\sigma$ is in a chiral superfield $\Sigma$ and the curvature is in a chiral superfield $\mathcal{R} = \bar{D}^2 \Sigma$. 

(2,2) SCFT
The supersymmetrization of the anomaly
\[
\delta_{\sigma} \log Z \sim \int \sqrt{\mathbb{V}} \, \delta \sigma \left( c \, R + g_{i\bar{i}} \partial_{\mu} \lambda^{i} \partial^{\mu} \bar{\lambda}^{\bar{i}} + \tilde{g}_{a\bar{a}} \partial_{\mu} \tilde{\lambda}^{a} \partial^{\mu} \bar{\tilde{\lambda}}^{\bar{a}} \right)
\]
after gauge fixing is
\[
\delta_{\Sigma} \log Z \sim \int d^{4} \theta \, (\delta \Sigma + \delta \bar{\Sigma}) \left( c \, (\Sigma + \bar{\Sigma}) + K_{c} (\lambda, \bar{\lambda}) - K_{tc} (\tilde{\lambda}, \bar{\tilde{\lambda}}) \right)
\]
The first term can also be written as \( c \int d^{2} \theta \, \delta \Sigma \, R + c. \, c. \).

- Lack of “B-anomaly” proves that \( K \) is such a sum of two terms.

- Invariance under Kahler transformations of \( K_{tc} (\tilde{\lambda}, \bar{\tilde{\lambda}}) \) by \( \bar{f} (\tilde{\lambda}) \) and of \( K_{c} (\lambda, \bar{\lambda}) \) by \( f (\lambda) \) up to a variation of a local counter term
\[
\int d^{4} \theta \delta \Sigma \, \bar{f} (\tilde{\lambda}) = \int d^{2} \theta \, \delta R \, f (\lambda).
\]

- Alternatively, Kahler invariance, involves a shift \( \Sigma \to \Sigma + \frac{1}{c} f (\lambda) \).
(2,2) SCFT

With an axial R-symmetry the supersymmetry current multiplet $J_{\pm \pm}$ is a real superfield satisfying (slightly simplified)

$$\bar{D}_{\mp} J_{\pm \pm} = \mp D_{\pm} \mathcal{W}$$

with chiral $\mathcal{W}$.

In an SCFT $\mathcal{W} = 0$ at separated points, but the anomaly sets a contact term

$$\mathcal{W} = \bar{D}^2 \left( c \Sigma + K_c (\lambda, \bar{\lambda}) - K_{tc} (\tilde{\lambda}, \tilde{\bar{\lambda}}) \right) =$$

$$= c \mathcal{R} + \bar{D}^2 \left( K_c (\lambda, \bar{\lambda}) - K_{tc} (\tilde{\lambda}, \tilde{\bar{\lambda}}) \right)$$

- It is invariant under Kahler transformations provided we also shift $\Sigma \rightarrow \Sigma + \frac{1}{c} f (\lambda)$. 
Ambiguities

• In the superconformal gauge the local terms are expressed in terms of the chiral curvature superfield $\mathcal{R} = \overline{D}^2 \Sigma$.

• Terms that depend on $\Sigma$ not through $\mathcal{R}$ are non-local.

• Therefore, the anomaly is not a variation of a local term.

• Freedom in the local term

$$\int d^2 \theta \ R \ f(\lambda) + c.c. = \int d^4 \theta \ \overline{\Sigma} f(\lambda) + c.c.$$ 

with holomorphic $f(\lambda)$ leads to freedom in Kahler transformations of $K_c(\lambda, \overline{\lambda})$. (It can be absorbed in a shift of $\Sigma$.)

• Other than that, the anomaly is unambiguous.
The anomaly in components

For a purely conformal variation $\delta \Sigma = \delta \sigma$ the anomaly is

$$\delta \Sigma \log Z \sim \int d^4 \theta \left( \delta \Sigma + \delta \Sigma \right) \left( c (\Sigma + \Sigma) + K_c (\lambda, \bar{\lambda}) - K_{tc} (\bar{\lambda}, \bar{\lambda}) \right)$$

$$= \int \left[ \delta \sigma \left( c \Box \sigma + g_{ii} \partial_{\mu} \lambda^i \partial_{\mu} \bar{\lambda}^i + \tilde{g}_{\alpha \bar{\alpha}} \partial_{\mu} \tilde{\lambda}^\alpha \partial_{\mu} \tilde{\lambda}^{\bar{\alpha}} \right) - \Box \delta \sigma K_c (\lambda, \bar{\lambda}) \right]$$

The last term leads to

$$\log Z \sim \int \sqrt{\gamma} R K_c (\lambda, \bar{\lambda}) + \ldots$$

and hence on $S^2$

$$Z = r^{c/3} e^{-K_c}$$

Q.E.D.
The anomaly in components

For a purely conformal variation $\delta \Sigma = \delta \sigma$ the anomaly is

$$\delta \Sigma \log Z \sim \int d^4 \theta \left( \delta \Sigma + \delta \bar{\Sigma} \right) \left( c \left( \Sigma + \bar{\Sigma} \right) + K_c(\lambda, \bar{\lambda}) - K_{tc}(\bar{\lambda}, \bar{\lambda}) \right)$$

$$= \int \left[ \delta \sigma \left( c \Box \sigma + g_{i\bar{i}} \partial_\mu \lambda^i \partial_\mu \bar{\lambda}^{\bar{i}} + \tilde{g}_{a\bar{a}} \partial_\mu \tilde{\lambda}^a \partial_\mu \bar{\tilde{\lambda}}^{\bar{a}} \right) \right.$$

$$- \Box \delta \sigma K_c(\lambda, \bar{\lambda}) \left. \right]$$

The last term leads to

$$\log Z \sim \int \sqrt{\gamma} R K_c(\lambda, \bar{\lambda}) + \ldots$$

Without SUSY this term is not universal. It is the variation of the local counterterm $\delta \Sigma$. With SUSY it is related to the universal term and hence it is meaningful.
Kahler invariance

\[ Z = r^{c/3} e^{-K_c} \]

- Kahler transformations of \( K_c \) can be absorbed in a local counter term.
- Alternatively, full Kahler invariance when \( \Sigma \) is also shifted. With this interpretation \( Z \) is invariant under the combined transformation

\[ K_c \rightarrow K_c + f(\lambda) + \bar{f}(\bar{\lambda}) \]

\[ r \rightarrow r e^{\frac{3}{c}(f(\lambda)+\bar{f}(\bar{\lambda}))} \]
Extensions

• Trivial to repeat with $\tilde{\Sigma}$ and to find $K_{tc}$.
  – Here we choose the contact terms to preserve the vector R-symmetry – use the other (2,2) supergravity.
  – Equivalently, we use a different regulator that preserves the supergravity of $\tilde{\Sigma}$.

• $2d$ $\mathcal{N} = (0,2)$
• $4d$
  – $\mathcal{N} = 1$
  – $\mathcal{N} = 2$
• The conformal manifold is parameterized by chiral superfields $\lambda^i$.
• The conformal factor $\sigma$ is in a chiral superfield $\Sigma$ and the curvature is in a chiral superfield $\mathcal{R}_- = \partial_{--}\bar{D}_+\bar{\Sigma}$.
• Several possible anomalies including:
  – The ordinary anomaly
    $$i\int d\theta^+ c \delta\Sigma \mathcal{R}_- + c.c. = i\int d^2\theta^+ c \delta\Sigma \partial_{--}\bar{\Sigma} + c.c.$$  
  – The anomaly associated with the metric
    $$i\int d^2\theta^+ (\delta\Sigma + \delta\bar{\Sigma})(K_i \partial_{--}\lambda^i - K_i \partial_{--}\bar{\lambda}^i)$$  
  – ...
\[ 2d \mathcal{N} = (0,2) \]

– The anomaly associated with the metric
\[ i \int d^2 \theta^+ \left( \delta \Sigma + \delta \bar{\Sigma} \right) \left( K_i \partial_{--} \lambda^i - K_{\bar{i}} \partial_{--} \bar{\lambda}^{\bar{i}} \right) \]

– Using the lack of “B–anomaly”
\[ i \int d^2 \theta^+ \left( \delta \Sigma + \delta \bar{\Sigma} \right) \left( K_i \partial_{--} \lambda^i - K_{\bar{i}} \partial_{--} \bar{\lambda}^{\bar{i}} \right) = i \int d^2 \theta^+ \left( \delta \Sigma + \delta \bar{\Sigma} \right) \left( \partial_i K \partial_{--} \lambda^i - \partial_{\bar{i}} K \partial_{--} \bar{\lambda}^{\bar{i}} \right) \]

with real \( K \).

– Hence, the metric on the conformal manifold must be Kahler.

– The last term includes in its component expansion \( \square \delta \sigma K \) and could lead to an interesting \( Z \). But…
Ambiguities in $2d\ \mathcal{N} = (0,2)$

- As in all our examples, a local counterterm
  \[ i\int d\theta^+ \mathcal{R} f(\lambda) + c.c. = i\int d^2\theta^+ \partial_-\bar{\Sigma} f(\lambda) + c.c. \]
  with holomorphic $f(\lambda)$ – Kahler transformations of $K$.

- Can redefine the 2d metric by a function of the moduli.
  - Locally a chiral superfield is the same as a real superfield (not in (2,2)). Hence, in the conformal gauge we can shift (more carefully, use SUGRA)
    \[ \Sigma \to \Sigma + \frac{i}{\partial_+} \bar{D}D_H(\lambda, \bar{\lambda}) . \]
    - This shifts the ordinary anomaly term
      \[ i\int d^2\theta^+ \delta\Sigma \partial_-\bar{\Sigma} \to i\int d^2\theta^+ \delta\Sigma \partial_-\bar{\Sigma} + i\int d^2\theta^+ \delta\Sigma \partial_-\bar{H} , \]
      which includes in components $\Box\delta\sigma H$.
  - Hence, $Z$ is ambiguous (note, there is no local counterterm).
4d

- Without SUSY
  - Anomaly $\int \sqrt{\gamma} \, \delta \sigma (a E_4 - c W^2 + g_{ij} \Box \lambda^i \Box \lambda^j + \cdots)$
  - Ambiguous counterterms
    $\int \sqrt{\gamma} \left( R^2 F_1(\lambda) + R_{\mu\nu}^2 F_2(\lambda) + R_{\mu\nu\rho\sigma}^2 F_3(\lambda) + \cdots \right)$

- With SUSY need to
  - Supersymmetrize – in $\mathcal{N} = 1, 2$ $\sigma$ and $\lambda$ in chiral superfields.
  - Covariantize in spacetime
  - Covariantize in the conformal manifold
4d $\mathcal{N} = 1$

- Anomaly
  \[ \int \sqrt{\gamma} \delta \sigma (a E_4 - c W^2 + g_{i \bar{i}} \lambda^i \bar{\lambda}^{\bar{i}} + \cdots) \]

- Ambiguous counterterms
  \[ \int \sqrt{\gamma} \left( R^2 F_1(\lambda, \bar{\lambda}) + R_{\mu \nu}^2 F_2(\lambda, \bar{\lambda}) + R_{\mu \nu \rho \sigma}^2 F_3(\lambda, \bar{\lambda}) + \cdots \right) \]

- Can be supersymmetrized and then the local counterterm makes the sphere partition function ambiguous [Gerchkovitz, Gomis, Komargodski].
$4d \mathcal{N} = 2$

- Example: theories of class S with moduli $\lambda$

- After a lot of algebra (using relevant formulas in the literature) the expressions without SUSY are supersymmetrized and covariantized to

$$\int \sqrt{\gamma} \delta \sigma \left( a E_4 + g_{i \bar{i}} \lambda^i \square \bar{\lambda}^{\bar{i}} + \cdots \right) + \sqrt{\gamma} K( \Box^2 \delta \sigma + \cdots)$$

Universal

Without SUSY this is not universal, but SUSY relates it to this and hence it is universal.
Key fact,

– In $\mathcal{N}=1$ the counterterm that is proportional to the curvature square is an arbitrary function of $\lambda$ and $\bar{\lambda}$.

– In $\mathcal{N}=2$ the counterterm that is proportional to the curvature square must be holomorphic in $\lambda$.

– Therefore, here the ambiguity is only in a holomorphic function of $\lambda$.

– Only Kahler transformations of $K$. 

4d $\mathcal{N} = 2$

• Collecting all the terms $\log Z \sim \int \sqrt{\gamma} E_4 K + \cdots$ and

\[ Z = r^{-a} e^{K(\lambda, \bar{\lambda})/12} \]

Ordinary conformal anomaly  The more subtle anomaly

• This reproduces a result of [Gerchkovitz, Gomis, Komargodski; Gomis, Ishtiaque]
Conclusions

• Anomaly under conformal transformations when the coupling constants depend on position
  – Unrelated to supersymmetry
  – This is a UV phenomenon
  • Visible on flat $\mathbb{R}^d$
  • Independent of the background

• Supersymmetry restricts
  – the form of the anomaly
  – the ambiguity due to local counterterms
Conclusions

• In 2d $\mathcal{N} = (2,2)$ the $S^2$ partition function depends on the anomaly.
  – New derivation of
    $$Z = r^{c/3} e^{-Kc}$$
    • The usual conformal anomaly
    • The more subtle conformal anomaly
  – Addresses the questions/confusions we raised...
Conclusions

• Addresses the questions/confusions we raised
  • One point function nonzero because of a term in the operator proportional to the curvature (like in the dilaton)
  • $Z$ can be unambiguous when the counterterm proportional to the curvature depends holomorphically on the couplings.
• Dependence on $\Sigma$ due to an anomaly. Different choices lead to $K_c$ or $K_{tc}$.
• The anomaly is set in the UV and is detected by the sphere (IR).
Conclusions

• Three step process
  – the anomaly
  – the ambiguity (freedom in counterterms)
  – the sphere

• Other cases
  – $2d \mathcal{N} = 1$ ambiguous
  – $2d \mathcal{N} = (0,2)$ ambiguous
  – $4d \mathcal{N} = 1$ ambiguous
  – $4d \mathcal{N} = 2$ $Z = r^{-a} e^{K/12}$